Deformations of equations and structure in nonlinear problems of mathematical physics

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Chapter 1

Introduction

Many processes in Nature can be described by using partial differential equations (PDEs). For instance, heat transfer is modelled by using the heat equation. In its simplest form, one denotes by $h(x,t)$ the temperature at a point x and a time t. The heat equation $h_t = h_{xx}$ balances the rate of change of $h(x,t)$ with respect to the time t and the change in the rate of change with respect to the position x. This equation is an example of a linear PDE but there are many phenomena that require nonlinear equations. It was a great discovery around 1967–68 that some classes of nonlinear PDEs can be solved effectively. The Korteweg–de Vries equation (describing waves in shallow water) provides a well-studied example of exactly solvable nonlinear PDE. Such equations are very important and the range of their applicability is exceptionally wide (e.g., they describe waves in a canal, tsunami's, propagation of light in nonlinear optics, and much more). For PDEs that encode processes in Nature, knowledge of physical conservation laws and symmetries is important for study of their properties. This interrelation between physics and mathematics leads to a beautiful and exciting area of research, to which this thesis intends to contribute. A crucial idea in this thesis is the concept of smooth deformation. Via synthesis of old and new geometric techniques we resolve Mathieu's problem, which was a long-standing open problem in the geometry of exactly solvable nonlinear PDEs.

The geometry of PDEs was born in the seminal works by Lie [89, 90], Bäcklund [5], Monge [104], Darboux [27], Bianchi [11], and E. Cartan [20]. Cartan's ideas were pursued by Spencer and his school [44, 117] in the 1960's. In essence, this branch if geometry aims to put PDEs and their solutions in a geometrical framework. For example, an equation such as $h_t = h_{xx}$ is interpreted as a hypersurface in 6-dimensional space with coordinates $(x, t, h, h_t, h_x, h_{xx})$ and some additional structures related to the derivatives. Ehresmann introduced the definition of jet spaces in [36]. This is an infinite-dimensional analogue of the above idea, with coordinates corresponding to all higher derivatives $\partial^k h/\partial x^k$. This concept was developed later by Ovsiannikov [107] and others [48, 118]. The discovery of integrability [102] and Hamiltonian interpretation of integrability [91] (see also [37, 127]) were a great contribution to the geometry of PDEs. Another crucial fact was the understanding that integrable systems admit infinite series of higher symmetries [12, 79, 106]. A symmetry of a given PDE is a particular type of map sending solutions of the PDE to solutions. For example, any constant c yields the symmetry $h \mapsto h + c$ of the heat equation $h_t = h_{xx}$. Zakharov and Shabat constructed a new method for solving nonlinear PDE in [128] based on the core idea of Gel'fand, Levitan, and Marchenko. Later Ablowitz, Kaup, Newell, and Segur generalised this approach [1].

The deformation approach is an important tool in the study of PDE. Gardner's deformations are an example of such concept. By definition, Gardner's deformation is a family of pairs consisting of deformation of equation and Miura's map which takes solutions of deformed equation to solutions of original equation. Using Gardner's deformation, one can recover recurrence relation between the Hamiltonians of a given PDE.

This thesis is devoted to solution of Mathieu's deformation problem for the N=2 supersymmetric a=4 Korteweg–de Vries equation (SKdV); the problem is to find integrable deformation for the N=2, a=4-SKdV that would reproduce its conservation laws. This problem was formulated by Mathieu in 1991 in [84]. Various attempts to solve it were undertaken since then but progress was limited.

The main results of the thesis are as follows:

1. There is no supersymmetry-invariant scaling-homogeneous polynomial Gardner's deformation for the N=2 supersymmetric a=4 Korteweg–de Vries equation such that under reduction this deformation would contain the classical Gardner deformation for the Korteweg–de Vries equation.

2. The super-Hamiltonians of the N=2 supersymmetric a=4 Korteweg–de Vries equation can be derived from the Hamiltonians of the Kaup–Boussinesq equation, and there is an explicit procedure for doing that.

3. For the Kaup–Boussinesq equation, there is a non-trivial Gardner's deformation that retracts under reduction to the classical Gardner formulas for the Korteweg–de Vries equation.

Secondly, we track the relations between zero-curvature representation for PDEs and Gardner's deformations in $\mathbb{Z}_2$-graded setup. This allows us to construct the second solution of Mathieu's deformation problem for the N=2, a=4-SKdV equation.

4. Marvan's technique for inspection of nonremovablity of spectral parameter in Lie algebra-valued zero-curvature representations for partial differential equations is generalised to the $\mathbb{Z}_2$-graded case.

5. The parameter in zero-curvature representation found by Das *et al.* for the N=2 supersymmetric a=4 Korteweg–de Vries equation is nonremovable. This zero-curvature representaion yields a non-trivial Gardner's deformation for the N=2, a=4-SKdV equation.

6. The zero-curvature representation found by Karasu–Kalkanlı, Sakovich and Yurduşen determines a non-trivial Gardner's deformation for Krasil'shchik–Kersten system.

7. Zero-curvature representations give rise to a natural class of non-Abelian variational Lie algebroids.

This thesis is devoted to the Korteweg–de Vries equation,

$$u_{12;t} = -u_{12;xxx} - 6u_{12}u_{12;x}, \tag{1.1}$$

and its generalizations [102]. We consider the completely integrable, multi-Hamiltonian evolutionary $N{=}2$ supersymmetric Korteweg–de Vries equation

$$\boldsymbol{u}_t = -\boldsymbol{u}_{xxx} + 3\big(\boldsymbol{u}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}\big)_x + \frac{a-1}{2}\big(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}^2\big)_x + 3a\boldsymbol{u}^2\boldsymbol{u}_x, \qquad \mathcal{D}_i = \frac{\vec{\partial}}{\partial\theta_i} + \theta_i\cdot\frac{\mathrm{d}}{\mathrm{d}x}, \tag{1.2}$$

for a scalar, complex bosonic $N{=}2$ superfield

$$\boldsymbol{u}(x,t;\theta_1,\theta_2) = u_0(x,t) + \theta_1\cdot u_1(x,t) + \theta_2\cdot u_2(x,t) + \theta_1\theta_2\cdot u_{12}(x,t),$$

where θ_1 and θ_2 are Grassmann variables satisfying $\theta_1^2 = \theta_2^2 = \theta_1\theta_2 + \theta_2\theta_1 = 0$.

The SKdV equation is most interesting (in particular, bi-Hamiltonian, whence completely integrable) if $a \in \{-2, 1, 4\}$, see [60, 84]. Let us consider *the bosonic limit* (or bosonic part),

$$u_1 = u_2 \equiv 0, \tag{1.3}$$

of system $N{=}2$ SKdV equation in components (3.5): by setting $a = -2$ we obtain the triangular system which consists of the modified KdV equation upon u_0 and the equation of KdV-type; in the case $a = 1$ we obtain the Krasil'shchik–Kersten system; for $a = 4$, we obtain a higher symmetry of the Kaup–Boussinesq equation,

$$u_{0;\xi} = \big(-u_{12} + 2u_0^2\big)_x, \qquad u_{12;\xi} = \big(u_{0;xx} + 4u_0u_{12}\big)_x. \tag{1.4}$$

Let us briefly list the content of chapters in this thesis.

In Chapter 2 we study Gardner's deformations for classical (non-graded) equations. We construct new Gardner's deformation of the Kaup–Boussinesq equation; the new family contains Gardner's deformation of KdV equation under reduction. Using this new Gardner's deformation for the Kaup–Boussinesq equation, we obtain recurrence formulas for the Hamiltonians of this equation. We prove that this new Gardner's deformation is non-trivial, i.e., it generates infinitely many non-trivial Hamiltonians. Indeed, we prove that every second Hamiltonian obtained by these formulas is non-trivial. We show also that Gardner's deformations can be considered as an initial point for construction of new integrable systems. In particular, we derive the Kaup–Newell equation from the Gardner deformation for the Kaup–Boussinesq equation.

In Chapter 3 we consider Gardner's deformations for supersymmetric equations. We present our first solution of Gardner deformation problem formulated by Mathieu for the $N{=}2$ supersymmetric $a{=}4$ Korteweg–de Vries equation (SKdV). On the one hand, we prove non-existence of supersymmetry-invariant scaling-homogeneous polynomial Gardner's deformation for the $N{=}2$, $a{=}4$-SKdV equation in super-fields such that the deformation would retract to Gardner's formulas for the KdV equation. On the other hand, we propose a two-step scheme for recursive production of integrals of motion for the $N{=}2$, $a{=}4$-SKdV. New Gardner's deformation of the Kaup–Boussinesq equation, which is contained in the bosonic limit of the super-hierarchy, yields the recurrence relation between the Hamiltonians of the limit, whence we determine the bosonic super-Hamiltonians of the full $N{=}2$, $a{=}4$-SKdV hierarchy.

Chapter 4 is devoted to Lie algebra-valued zero-curvature representations for $\mathbb{Z}_2$-graded partial differential equations (PDE). We generalise to $\mathbb{Z}_2$-graded setup Marvan's technique for inspection of nonremovability of spectral parameter in zero-curvature representations. We prove that cohomological interpretation of this result works also in the $\mathbb{Z}_2$-graded case. We prove that the parameter in zero-curvature representation for $N{=}2$, $a{=}4$-SKdV found by Das *et al.* is nonremovable.

In Chapter 5 we consider Gardner's deformations and zero-curvature representation in context of differential covering over PDE (known also as systems of nonlocalities). We stress that Gardner's deformations, zero-curvature representations, and covering over PDE are different realisations of one object. We illustrate a link between deformation techniques of two types, namely, Marvan's technique for inspection of nonremovability of spectral parameters in zero-curvature representations and Frölicher–Nijenhuis bracket formalism developed by Krasil'shchik *et al.* Using this relation between zero-curvature representations and Gardner's deformations, we construct the second solution of deformation problem for the $N{=}2$, $a{=}4$-SKdV equation. We prove that the zero-curvature representation found by Das *et al.* yields the covering (i.e., a system of nonlocalities) that under reduction contains Gardner's deformation for the KdV equation as well as Gardner's deformation for the $N{=}1$ supersymmetric Korteweg–de Vries equation (sKdV). We study the (non)removability of parameters under gauge and other types of transformations in matrix zero-curvature representations, as well as in nonlocal structures which were introduced by Gardner or Sasaki for the classical Korteweg–de Vries equation and which were constructed by Das *et al.* [28] for the $N{=}2$, $a{=}4$-SKdV equation.

In Chapter 6 we show that zero-curvature representation for PDE give rise to a natural class of non-Abelian variational Lie algebroids. We realise non-Abelian variational Lie algebroids via BRST-like homological Hamiltonian vector field on superbundles. This relates the research to the geometry of quantum inverse scattering. Let us recall here that Berezin developed new super-mathematics with an eye towards its applications in quantum mechanics and statistical physics [10, 86, 87].

The results obtained in this thesis are related to research of the following scientific schools. Recently, Krasil'shchik *et al.* developed cohomological deformation theory that describes infinitesimal behaviour of families of nonlocalites over PDEs. We use this approach intensively in Chapter 5. Independently from each other, Marvan and Sakovich developed a method for inspection of (non)removability of parameters in zero-curvature representations for PDEs. We generalise this method to the $\mathbb{Z}_2$-graded case in Chapter 4 and we use this generalisation for solving Gardner's deformation problem for the N=2, a=4-SKdV equation. Thirdly, Dubrovin *et al.* and Ferapontov developed novel approaches for construction of new integrable systems on the basis of known ones, focusing also on the global structure of the space of integrable systems. In Chapter 2 we describe a regular way of obtaining initial data for those approaches and we comment on the adjacency relations between integrable systems in the arising moduli spaces. Finally, we recall that Faddeev and his school contributed fundamentally to the concept of quantum inverse scattering method; we approach their scheme in Chapter 6. The results of this thesis may also be interesting to specialists in supersymmetry (e.g. supergravity) and to experts in superintegrability in the sense of Winternitz.

The chapters of this thesis are based directly on recent peer-review articles and one preprint.

- Chapter 2 is based on the articles [JMP10, JPCS14].
- Chapter 3 is based on the article [JMP10].
- Chapter 4 is based on the preprint [1301.7143].
- Chapter 5 is based on the article [JMP12] and preprint [1301.7143].
- Chapter 6 is based on the article [JNMP14].

[JMP10] *Hussin V., Kiselev A. V., Krutov A. O., Wolf T.* (2010) N=2 supersymmetric a=4-KdV hierarchy derived via Gardner's deformation of Kaup–Boussinesq equation, *J. Math. Phys.* **51**:8, 083507, 19 p. `arXiv:0911.2681` [nlin.SI]

[JMP12] *Kiselev A. V., Krutov A. O.* (2012) Gardner's deformations of the graded Korteweg–de Vries equations revisited, *J. Math. Phys.* **53**:10, 103511, 18 p. `arXiv:1108.2211` [nlin.SI]

[JNMP14] *Kiselev A. V., Krutov A. O.* (2014) Non-Abelian Lie algebroids over jet spaces, *J. Nonlin. Math. Phys.* **21**:2, 188-213. `arXiv:1305.4598` [math.DG]

[JPCS14] *Kiselev A. V., Krutov A. O.* (2014) Gardner's deformations as generators of new integrable systems, *J. Phys. Conf. Ser.* **482**, Proc. Int. workshop 'Physics and Mathematics of Nonlinear Phenomena' (June 22–29, 2013; Gallipoli (LE), Italy), 012021, 6 p. `arXiv:1312.6941` [nlin.SI]

[1301.7143] *Kiselev A. V., Krutov A. O.* (2014) On the (non)removability of spectral parameters in $\mathbb{Z}_2$-graded zero-curvature representations and its applications. – 22 p. *Preprint* `arXiv:1301.7143v2` [math.DG]

Chapter 2

Gardner's deformations of non-graded equations

Let us first briefly recall some definitions (see [12, 62, 106] and [94] for detail); this material is standard so we now fix the notation.

2.1 The geometry of infinite jet space $J^\infty(\pi)$

Let M^n be a smooth real n-dimensional orientable manifold. Consider a smooth vector bundle $\pi\colon E^{n+m} \to M^n$ with m-dimensional fibres and construct the space $J^\infty(\pi)$ of infinite jets of sections for π. Let $\mu^k_{\boldsymbol{x}_0}$ be a set of local sections $s \in \Gamma(\pi)$ such that s is vanish in $\boldsymbol{x}_0 \in M^n$ together with its derivatives with order less or equal k:

$$\mu^k_{\boldsymbol{x}_0} = \{s \in \Gamma(\pi) \mid \exists r \in \Gamma(\pi)\colon s = (\boldsymbol{x} - \boldsymbol{x}_0) \cdot r\}\,.$$

Consider the equivalence classes of (local) sections at a point $\boldsymbol{x}_0$

$$J^k_{\boldsymbol{x}_0}(\pi) = \Gamma(\pi)/\mu^k_{\boldsymbol{x}_0}.$$

The jet space $J^k(\pi)$ of k-th jets of sections for the vector bundle π is the union

$$J^k(\pi) = \bigcup_{\boldsymbol{x}_0 \in M^n} J^k_{\boldsymbol{x}_0}(\pi).$$

The infinite jet space $J^\infty(\pi)$ is the projective limit,

$$J^\infty(\pi) = \varprojlim_{k\to+\infty} J^k(\pi).$$

A convenient organization of local coordinates is as follows: let x^i be some coordinate system on a chart in the base M^n and denote by u^j the coordinates along a fibre of the bundle π so that the variables u^j play the rôle of unknowns; one obtains the collection u^j_σ of jet variables along fibres of the vector bundle $J^\infty(\pi) \to M^n$ (here $|\sigma| \geqslant 0$ and $u^j_\varnothing \equiv u^j$). (In particular, we have $n = 2$, $m = 1$, $x^1 = x$, $x^2 = t$, $u^1 = u_{12}$ for the KdV equation (1.1) and $n = 2$, $m = 2$, $x^1 = x$, $x^2 = \xi$, $u^1 = u_{12}$, $u^2 = u_0$ for the Kaup–Boussinesq equation (1.4).)

We define the ring of smooth function on $J^\infty(\pi)$ as inductive limit

$$C^\infty(J^\infty(\pi)) = \bigl\{f\colon J^\infty(\pi) \to \mathbb{R} \mid \exists k \in \mathbb{N} \text{ such that } f \in C^\infty(J^k(\pi))\bigr\}.$$

In this setup, the *total derivatives* D_{x^i} are commuting vector fields

$$D_{x^i} = \tfrac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma i}\, \tfrac{\partial}{\partial u^j_\sigma}$$

on $J^\infty(\pi)$. We also denote total derivatives by $\frac{\mathrm{d}}{\mathrm{d}x^i}$. We will use both notations and will not make any distinction between them.

Consider a system of partial differential equations

$$\mathcal{E} = \bigl\{F^\ell(x^i, u^j, \ldots, u^j_\sigma, \ldots) = 0, \quad \ell = 1, \ldots, r < \infty\bigr\};$$

without any loss of generality for applications we assume that the system at hand satisfies mild assumptions which are outlined in [62, 106]. Then the system $\mathcal{E}$ and all its differential consequences $D_\sigma(F^\ell) = 0$ (thus presumed existing, regular, and not leading to any contradiction in the course of derivation) generate the infinite prolongation $\mathcal{E}^\infty$ of the system $\mathcal{E}$.

Let us denote by $\bar{D}_{x^i}$ the restrictions of total derivatives D_{x^i} to $\mathcal{E}^\infty \subseteq J^\infty(\pi)$. We recall that the vector fields $\bar{D}_{x^i}$ span the Cartan distribution $\mathcal{C}$ in the tangent space $T\mathcal{E}^\infty$. At every point $\theta^\infty \in \mathcal{E}^\infty$ the tangent space $T_{\theta^\infty}\mathcal{E}^\infty$ splits in a direct sum of two subspaces. The one which is spanned by the Cartan distribution $\mathcal{E}^\infty$ is *horizontal* and the other is *vertical*: $T_{\theta^\infty}\mathcal{E}^\infty = \mathcal{C}_{\theta^\infty} \oplus V_{\theta^\infty}\mathcal{E}^\infty$. We denote by $\Lambda^{1,0}(\mathcal{E}^\infty) = \operatorname{Ann}\mathcal{C}$ and $\Lambda^{0,1}(\mathcal{E}^\infty) = \operatorname{Ann} V\mathcal{E}^\infty$ the $C^\infty(\mathcal{E}^\infty)$-modules of contact and horizontal one-forms which vanish on $\mathcal{C}$ and $V\mathcal{E}^\infty$, respectively. Denote further by $\Lambda^r(\mathcal{E}^\infty)$ the $C^\infty(\mathcal{E}^\infty)$-module of r-forms on $\mathcal{E}^\infty$. There is a natural decomposition $\Lambda^r(\mathcal{E}^\infty) = \bigoplus_{q+p=r} \Lambda^{p,q}(\mathcal{E}^\infty)$, where $\Lambda^{p,q}(\mathcal{E}^\infty) = \bigwedge^p \Lambda^{1,0}(\mathcal{E}^\infty) \wedge \bigwedge^q \Lambda^{0,1}(\mathcal{E}^\infty)$. This implies that the de Rham differential $\bar{\mathrm{d}}$ on $\mathcal{E}^\infty$ is subjected to the decomposition $\bar{\mathrm{d}} = \bar{\mathrm{d}}_h + \bar{\mathrm{d}}_\mathcal{C}$, where $\bar{\mathrm{d}}_h\colon \Lambda^{p,q}(\mathcal{E}^\infty) \to \Lambda^{p,q+1}(\mathcal{E}^\infty)$ is the horizontal differential and $\bar{\mathrm{d}}_\mathcal{C}\colon \Lambda^{p,q}(\mathcal{E}^\infty) \to \Lambda^{p+1,q}(\mathcal{E}^\infty)$ is the vertical differential. In local coordinates, the differential $\bar{\mathrm{d}}_h$ acts by the rule

$$\bar{\mathrm{d}}_h = \sum\nolimits_i \mathrm{d}x^i \wedge \bar{D}_{x^i}.$$

We will use this formula in what follows. By definition, we put $\bar{\Lambda}(\mathcal{E}^\infty) = \bigoplus_{q\geqslant 0} \Lambda^{0,q}(\mathcal{E}^\infty)$ and we denote by $\overline{H}^n(\cdot)$ the senior d_h-cohomology groups (also called senior *horizontal cohomology*) for the infinite jet bundles which are indicated in parentheses, cf. [63].

We denote $f(x^i, [u^j]) = f(x^1, \ldots, x^n, u^1, \ldots u^m, u^1_x, \ldots, u^j_\sigma)$ for $|\sigma| < k(f)$.

A conserved current η for the system $\mathcal{E}$ is the continuity equation

$$\sum_{i=1}^n \bar{D}_{x^i}(\eta_i) \doteq 0 \text{ on } \mathcal{E}^\infty,$$

where $\doteq$ denotes equality upon a system $\mathcal{E}$ and $\eta_i(x^i, [u^j])$ are the coefficients of the horizontal $(n-1)$-form,

$$\eta = \sum_{i=1}^{n} (-1)^{i+1} \eta_i \cdot \mathrm{d}x^1 \wedge \ldots \wedge \widehat{\mathrm{d}x^i} \wedge \ldots \wedge \mathrm{d}x^n \in \bar{\Lambda}^{n-1}(\pi),$$

where $\widehat{\mathrm{d}x^i}$ denotes omitted element. The coefficient η^1 is called *conserved density* and coefficients $\eta^2, \ldots, \eta^n$ are called *flux components.* The conservation of η is the equality

$$\bar{\mathrm{d}}_h\big|_{\mathcal{E}^\infty} \eta \doteq 0 \text{ on } \mathcal{E}^\infty.$$

A conservation law $\int \eta \in \overline{H}^n(\pi)$ for an equation $\mathcal{E}$ is the equivalence class of conserved currents η, modulo the globally defined exact currents $\bar{\mathrm{d}}\xi \in \int 0$ (i.e. two conservation laws η_1 and η_2 are equivalent if their difference is a exact form $\eta_1 - \eta_2 = \bar{\mathrm{d}}\xi$).

The vector field

$$\partial^{(u)}_{\varphi} = \sum_{j=1}^{m} \sum_{|\sigma| \geqslant 0} D_{x^\sigma}(\varphi) \frac{\partial}{\partial u^j_\sigma}$$

is the *evolutionary derivation* along the fibre of the infinite jet bundle $J^\infty(\pi) \to M^n$ over the vector bundle π with fibre variables u. The m-tuple $\varphi = {}^{\mathrm{t}}(\varphi^1, \ldots, \varphi^m) \in \Gamma(\pi) \otimes_{C^\infty(M^n)} C^\infty(J^\infty(\pi))$ is the generating section of $\partial^{(u)}_{\varphi}$. By construction, the *generating section* of φ is a section of the induced vector bundle $\pi^*_\infty(\pi) \colon E^{n+m} \times_{M^n} J^\infty(\pi) \to J^\infty(\pi)$; here we implicitly use the fact that $\pi \colon E^{n+m} \to M^n$ is a vector bundle and hence the tangent space at the points of its fibres are the fibres themselves (otherwise, the construction would be $\varphi \in \Gamma(\pi^*_\infty(T\pi))$ for a fibre bundle π). We denote $\varkappa(\pi) \equiv \Gamma(\pi^*_\infty(\pi))$ for brevity.

The restriction of evolutionary vector field $\partial^{(u)}_{\varphi}$ on $\mathcal{E}^\infty$ is called an infinitesimal symmetry (see [12, 42, 62]) of the equation $\mathcal{E} = \{F^l = 0\}_{l=1}^{r}$ if the determining equations

$$\partial^{(u)}_{\varphi}\big|_{\mathcal{E}^\infty} (F^l) \doteq 0, \quad l = 1, \ldots, r,$$

hold by virtue of equation $\mathcal{E}$.

2.2 Gardner's deformations

Definition 1 ([59, 82, 101])**.** Let $\mathcal{E} = \{u_t = f(x, [u])\}$ be a system of evolution equations (in particular, a completely integrable system). Suppose $\mathcal{E}(\varepsilon) = \{\tilde{u}_t = f_\varepsilon(x, [\tilde{u}], \varepsilon) \mid f_\varepsilon \in \operatorname{im} \frac{\mathrm{d}}{\mathrm{d}x}\}$ is a deformation of $\mathcal{E}$ such that at each point $\varepsilon \in \mathcal{I}$ of an interval $\mathcal{I} \subseteq \mathbb{R}$ there is the *Miura contraction* $\mathfrak{m}_\varepsilon = \{u = u([\tilde{u}], \varepsilon)\} \colon \mathcal{E}(\varepsilon) \to \mathcal{E}$. Then the pair $(\mathcal{E}(\varepsilon), \mathfrak{m}_\varepsilon)$ is the (*classical*) *Gardner deformation* for system $\mathcal{E}$.

Under the assumption that $\mathcal{E}(\varepsilon)$ be in the form of a (super-)conserved current, the Taylor coefficients $\tilde{u}^{(k)}$ of the formal power series $\tilde{u} = \sum_{k=0}^{+\infty} \tilde{u}^{(k)} \cdot \varepsilon^k$ are termwise conserved on $\mathcal{E}(\varepsilon)$ and hence on $\mathcal{E}$. Therefore, the contraction $\mathfrak{m}_\varepsilon$ yields the recurrence relations, ordered by the powers of ε, between these densities $\tilde{u}^{(k)}$, while the equality $\mathcal{E}(0) = \mathcal{E}$ specifies its initial condition.

Example 1 ([102])**.** The contraction

$$\mathfrak{m}_\varepsilon = \left\{ u_{12} = \tilde{u}_{12} \pm \varepsilon \tilde{u}_{12;x} - \varepsilon^2 \tilde{u}_{12}^2 \right\} \tag{2.1a}$$

maps solutions $\tilde{u}_{12}(x,t;\varepsilon)$ of the extended equation $\mathcal{E}(\varepsilon)$,

$$\tilde{u}_{12;t} + \left(\tilde{u}_{12;xx} + 3\tilde{u}_{12}^2 - 2\varepsilon^2 \cdot \tilde{u}_{12}^3 \right)_x = 0, \tag{2.1b}$$

to solutions $u_{12}(x,t)$ of Korteweg–de Vries equation (1.1). Plugging the series $\tilde{u}_{12} = \sum_{k=0}^{+\infty} u_{12}^{(k)} \cdot \varepsilon^k$ into the expression $\mathfrak{m}_\varepsilon$ for $\tilde{u}_{12}$, we obtain the chain of equations ordered by the powers of ε,

$$u_{12} = \sum_{k=0}^{+\infty} \tilde{u}_{12}^{(k)} \cdot \varepsilon^k \pm \tilde{u}_{12;x}^{(k)} \cdot \varepsilon^{k+1} - \sum_{\substack{i+j=k \\ i,j \geq 0}} \tilde{u}_{12}^{(i)} \tilde{u}_{12}^{(j)} \cdot \varepsilon^{k+2}.$$

Let us fix the plus sign in (2.1a) by reversing $\varepsilon \to -\varepsilon$ if necessary. Equating the coefficients of ε^k, we obtain the relations

$$u = \tilde{u}_{12}^{(0)}, \qquad 0 = \tilde{u}_{12}^{(1)} + \tilde{u}_{12;x}^{(0)}, \qquad 0 = \tilde{u}_{12}^{(k)} + \tilde{u}_{12;x}^{(k-1)} - \sum_{\substack{i+j=k-2 \\ i,j \geq 0}} \tilde{u}_{12}^{(i)} \tilde{u}_{12}^{(j)}, \quad k \geq 2.$$

Hence, from the initial condition $\tilde{u}_{12}^{(0)} = u_{12}$, we recursively generate the densities

$$\begin{aligned}
\tilde{u}_{12}^{(1)} &= -u_{12;x}, \qquad \tilde{u}_{12}^{(2)} = u_{12;xx} - u_{12}^2, \qquad \tilde{u}_{12}^{(3)} = -u_{12;xxx} + 4u_{12;x}u_{12}, \\
\tilde{u}_{12}^{(4)} &= u_{12;4x} - 6u_{12;xx}u_{12} - 5u_{12;x}^2 + 2u_{12}^3, \\
\tilde{u}_{12}^{(5)} &= -u_{12;5x} + 8u_{12;xxx}u_{12} + 18u_{12;xx}u_{12;x} - 16u_{12;x}u_{12}^2, \\
\tilde{u}_{12}^{(6)} &= u_{12;6x} - 10u_{12;4x}u_{12} - 28u_{12;xxx}u_{12;x} - 19u_{12;xx}^2 + 30u_{12;xx}u_{12}^2 + 50u_{12;x}^2u_{12} - 5u_{12}^4, \\
\tilde{u}_{12}^{(7)} &= -u_{12;7x} + 12u_{12;5x}u_{12} + 40u_{12;4x}u_{12;x} + 68u_{12;xxx}u_{12;xx} - 48u_{12;xxx}u_{12}^2 \\
&\quad - 216u_{12;xx}u_{12;x}u_{12} - 60u_{12;x}^3 + 64u_{12;x}u_{12}^3, \quad \textit{etc.}
\end{aligned}$$

The conservation $\tilde{u}_{12;t} = \frac{\mathrm{d}}{\mathrm{d}x}(\cdot)$ implies that each coefficient $u_{12}^{(k)}$ is conserved on (1.1).

The densities $u_{12}^{(2k)} = c(k) \cdot u_{12}^k + \ldots$, $c(k) = \mathrm{const}$, determine the Hamiltonians $\mathcal{H}_{12}^{(k)} = \int h_{12}^{(k)}[u_{12}] \, \mathrm{d}x$ of the renowned KdV hierarchy. Let us show that all of them are nontrivial.

Consider the zero-order part $\breve{u}_{12}^{\text{KdV}}$ such that $\tilde{u}_{12}\bigl([u_{12}],\varepsilon\bigr) = \breve{u}_{12}^{\text{KdV}}(u_{12},\varepsilon) + \ldots$, where the dots denote summands containing derivatives of u_{12}. Taking the zero-order component of (2.1a), we conclude that the generating function $\breve{u}_{12}^{\text{KdV}}$ satisfies the algebraic recurrence relation $u_{12} = \breve{u}_{12}^{\text{KdV}} - \varepsilon^2\bigl(\breve{u}_{12}^{\text{KdV}}\bigr)^2$. We choose the root by the initial condition $\breve{u}_{12}^{\text{KdV}}\bigr|_{\varepsilon=0} = u_{12}$, which yields

$$\breve{u}_{12}^{\text{KdV}} = \Bigl(1 - \sqrt{1-4\varepsilon^2 u_{12}}\Bigr)/(2\varepsilon^2). \tag{2.2}$$

Moreover, the Taylor coefficients $\breve{u}_{12}^{(k)}(u_{12})$ in $\breve{u}_{12}^{\text{KdV}} = \sum_{k=0}^{+\infty} \breve{u}_{12}^{(k)} \cdot \varepsilon^{2k}$ equal $c(k)\cdot u_{12}^{k+1}$, where $c(k)$ are positive and grow with k. This is readily seen by induction on k with the base $\breve{u}_{12}^{(0)} = u_{12}$. Expanding both sides of the equality $u_{12} = \breve{u}_{12}^{\text{KdV}} - \varepsilon^2\cdot\bigl(\breve{u}_{12}^{\text{KdV}}\bigr)^2$ in ε^2, we notice that

$$\breve{u}_{12}^{(k)} = \sum_{\substack{i+j=k-1,\\ i,j\geq 0}} \breve{u}_{12}^{(i)} \cdot \breve{u}_{12}^{(j)} = \sum_{i+j=k-1} c(i)c(j)\cdot u_{12}^{k+1}.$$

Therefore, the next coefficient, $c(k) = \sum_{i+j=k-1} c(i)\cdot c(j)$, is the sum over $i,j \geq 0$ of products of positive numbers, whence $c(k+1) > c(k) > 0$. This proves the claim.

Let us list the densities $h_{\text{KdV}}^{(k)} \sim u_{12}^{(2k)} \mod \operatorname{im} \mathrm{d}/\mathrm{d}x$ of the first seven Hamiltonians for (1.1). These will be correlated in section 3.5 with the lowest seven Hamiltonians for (1.2), see [84] and (3.12) below. We have

$$\begin{aligned}
&h_{\text{KdV}}^{(1)} = u_{12}^2, \qquad h_{\text{KdV}}^{(2)} = 2u_{12}^3 - u_{12;x}^2 + 2u_{12}^3 + u_{12;xx}, \qquad h_{\text{KdV}}^{(3)} = 5u_{12}^4 + 5u_{12;xx}u_{12}^2 + u_{12;xx}^2,\\
&h_{\text{KdV}}^{(4)} = -14u_{12}^5 + 70u_{12}^2u_{12;x}^2 + 14u_{12}u_{12;xxx}u_{12;x} + u_{12;xxx}^2,\\
&h_{\text{KdV}}^{(5)} = 42u_{12}^6 - 420u_{12}^3u_{12;x}^2 + 9u_{12}^2u_{12;6x} + 126u_{12}^2u_{12;xx}^2 + u_{12;4x}^2 - 7u_{12;xx}^3 - 35u_{12;x}^4,\\
&h_{\text{KdV}}^{(6)} = 1056u_{12}^7 - 18480u_{12}^4u_{12;x}^2 + 7392u_{12}^3u_{12;xx}^2 + 55u_{12}^2u_{12;8x} - 1584u_{12}^2u_{12;xxx}^2\\
&\qquad + 66u_{12}u_{12;4x}^2 + 3520u_{12}u_{12;xx}^3 - 6160u_{12}u_{12;x}^4 - 8u_{12;5x}^2 + 3696u_{12;xx}^2u_{12;x}^2,\\
&h_{\text{KdV}}^{(7)} = 15444u_{12}^8 - 432432u_{12}^5u_{12;x}^2 + 4004u_{12}^4u_{12;6x} + 216216u_{12}^4u_{12;xx}^2 + 2145u_{12}^3u_{12;8x}\\
&\qquad - 45760u_{12}^3u_{12;xxx}^2 + 3861u_{12}^2u_{12;4x}^2 + 133848u_{12}^2u_{12;xx}^3 - 360360u_{12}^2u_{12;x}^4\\
&\qquad - 936u_{12}u_{12;5x}^2 + 36u_{12;6x}^2 + 6552u_{12;4x}^2u_{12;xx} + 72072u_{12;xxx}^2u_{12;x}^2 - 28314u_{12;xx}^4.
\end{aligned}$$

At the same time, the densities $u_{12}^{(2k+1)} = \frac{\mathrm{d}}{\mathrm{d}x}\bigl(\cdot\bigr) \sim 0$ are trivial (i.e. for all $k \in \mathbb{N}$ exists $g_{2k+1} \in C^\infty(\mathcal{E}^\infty)$ such that $u_{12}^{(2k+1)} = \frac{\mathrm{d}}{\mathrm{d}x} g_{2k+1}$). Indeed, for $\omega_0 := \sum_{k=0}^{+\infty} u_{12}^{(2k)} \cdot \varepsilon^{2k}$ and $\omega_1 := \sum_{k=0}^{+\infty} u_{12}^{(2k+1)} \cdot \varepsilon^{2k}$ such that $\tilde{u} = \omega_0 + \varepsilon\cdot\omega_1$, we equate the odd powers of ε in (2.1a) and obtain $\omega_1 = \frac{1}{2\varepsilon^2}\frac{\mathrm{d}}{\mathrm{d}x}\log\bigl(1 - 2\varepsilon^2\omega_0\bigr)$.

In what follows, using deformation (2.1) of (1.1), we fix the coefficients of differential monomials in u_{12} within a bigger deformation problem (see section 2.3) for two-component system (3.10).

2.3 New deformation of the Kaup–Boussinesq equation

In this section we construct a new Gardner's deformation for the Kaup–Boussinesq equation (1.4), which is the bosonic limit of the $N{=}2$ supersymmetric system (3.8). We will use known deformation (2.1) to fix several coefficients in the Miura contraction $\mathfrak{m}_\varepsilon$, which ensures the difference of the new solution (2.3)–(2.4) from previously known deformations of (1.4), see [52]. We prove that the new deformation is maximally nontrivial: It yields infinitely many nontrivial conserved densities, and none of the Hamiltonians is lost.

Let us summarize well-known properties of the Kaup–Boussinesq equation [54, 56, 105]:

Proposition 1 ([56])**.** Completely integrable Kaup–Boussinesq system (1.4) is a tri-Hamiltonian equation

$$\begin{pmatrix} u_0 \\ u_{12} \end{pmatrix}_\xi = \hat{A}_1^{12} \begin{pmatrix} \delta/\delta u_0 \\ \delta/\delta u_{12} \end{pmatrix} \Big(\int \big[2u_0^2 u_{12} - \tfrac{1}{2} u_{12}^2 - \tfrac{1}{2} u_{0;x}^2 \big] \,\mathrm{d}x \Big)$$

$$= \hat{A}_1^{0} \begin{pmatrix} \delta/\delta u_0 \\ \delta/\delta u_{12} \end{pmatrix} \Big(- \int u_0 u_{12} \,\mathrm{d}x \Big) = \hat{A}_2 \begin{pmatrix} \delta/\delta u_0 \\ \delta/\delta u_{12} \end{pmatrix} \Big(- \int u_{12} \,\mathrm{d}x \Big).$$

The senior Hamiltonian operator $\hat{A}_2$ is

$$\begin{pmatrix} u_{0;x} + 2u_0 \frac{\mathrm{d}}{\mathrm{d}x} & u_{12;x} - 4u_0 u_{0;x} - 2u_0^2 \frac{\mathrm{d}}{\mathrm{d}x} + 2u_{12} \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 \\ u_{12;x} - 2u_0^2 \frac{\mathrm{d}}{\mathrm{d}x} + 2u_{12} \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 & -4u_0 u_{12} \frac{\mathrm{d}}{\mathrm{d}x} - 4 \frac{\mathrm{d}}{\mathrm{d}x} \circ u_0 u_{12} - u_0 \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 - \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 \circ u_0 \end{pmatrix}.$$

The junior Hamiltonian operators $\hat{A}_1^0$ and $\hat{A}_1^{12}$ are obtained from $\hat{A}_2$ by the shifts of the respective fields, c.f. [29, 115]:

$$\hat{A}_1^0 = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & -2u_{0;x} - 2u_0 \frac{\mathrm{d}}{\mathrm{d}x} \\ -2u_0 \frac{\mathrm{d}}{\mathrm{d}x} & -2u_{12;x} - 4u_{12} \frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 \end{pmatrix} = \frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \hat{A}_2\Big|_{u_0+\lambda}$$

and

$$\hat{A}_1^{12} = \begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}x} \\ \frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix} = \frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}\mu}\Big|_{\mu=0} \hat{A}_2\Big|_{u_{12}+\mu}.$$

The three operators $\hat{A}_1^0$, $\hat{A}_1^{12}$, and $\hat{A}_2$ are Poisson compatible (i.e. their linear combination $\lambda_1 \hat{A}_1^0 + \lambda_2 \hat{A}_1^{12} + \lambda_3 \hat{A}_2$ are Hamiltonian operator).

Kaup–Boussinesq equation (1.4) admits an infinite sequence of integrals of motion. We will derive them via the Gardner deformation. Unlike it was in [52], from now on we always assume that (2.1a) is recovered under $\tilde{u}_0 := 0$.

The Magri scheme for Kaup–Boussinesq equation is following

$$
\begin{tikzcd}
 & \cdots \arrow[dr, "A_1^0"] & & & \\
 & & \varphi_3 & t_3 \equiv t & \text{Bosonic limit of SKdV} \\
h_2 = u_{12}u_0 \arrow[r, "\delta"] & \psi_3 \arrow[ur, "A_2"] \arrow[dr, "A_1^0"] \arrow[dddr, "A_1^{12}"] & & & \\
 & & \varphi_2 & t_2 \equiv \xi & \text{Kaup–Boussinesq equation} \\
h_1 = u_{12} \arrow[r, "\delta"] & \psi_2 \arrow[ur, "A_2"] \arrow[dr, "A_1^0"] & & & \\
 & & \varphi_1 & t_1 \equiv x & \text{translation w.r.t. } x \\
h_0 = u_0 \arrow[r, "\delta"] & \psi_1 \arrow[ur, "A_2"] & & &
\end{tikzcd}
$$

We assume that both the extension $\mathcal{E}(\varepsilon)$ of (1.4) and the contraction $\mathfrak{m}_\varepsilon\colon \mathcal{E}(\varepsilon) \to \mathcal{E}$ into (1.4) are homogeneous polynomials in ε. From now on, we denote reduction (1.4) by $\mathcal{E}$.

First, let us estimate the degrees in ε for such polynomials $\mathcal{E}(\varepsilon)$ and $\mathfrak{m}_\varepsilon$, by balancing the powers of ε in the left- and right-hand sides of (1.4) with u_0 and u_{12} replaced by the Miura contraction $\mathfrak{m}_\varepsilon = \big\{u_0 = u_0\big([\tilde{u}_0, \tilde{u}_{12}], \varepsilon\big),\ u_{12} = u_{12}\big([\tilde{u}_0, \tilde{u}_{12}], \varepsilon\big)\big\}$. The time evolution in the left-hand side, which is of the form $u_\xi = \partial^{(u)}_{\tilde{u}_\xi}(\mathfrak{m}_\varepsilon)$ by the chain rule, sums the degrees in ε: $\deg u_\xi = \deg \mathfrak{m}_\varepsilon + \deg \mathcal{E}(\varepsilon)$. At the same time, we notice that system (1.4) is only quadratic-nonlinear. Hence its right-hand side, with $\mathfrak{m}_\varepsilon$ substituted for u_0 and u_{12}, gives the degree $2 \times \deg \mathfrak{m}_\varepsilon$, irrespective of $\deg \mathcal{E}(\varepsilon)$. Consequently, we obtain the balance[1] $1 : 1$ for $\max\deg \mathfrak{m}_\varepsilon : \max\deg \mathcal{E}(\varepsilon)$. This is in contrast with the balance $1 : 2$ for polynomial deformations of bosonic limit (3.10) for initial SKdV system (1.2), which is cubic-nonlinear[2] (c.f. [84]).

Obviously, a lower degree polynomial extension $\mathcal{E}(\varepsilon)$ contains fewer undetermined coefficients. This is the first profit we gain from passing to (3.8) instead of (1.2). By the same argument, we conclude that $\mathfrak{m}_\varepsilon \colon \mathcal{E}(\varepsilon) \to \mathcal{E}$, viewed as the algebraic system upon these coefficients, is only *quadratic*-nonlinear w.r.t. the coefficients in $\mathfrak{m}_\varepsilon$ (and, obviously, *linear* w.r.t. the coefficients in $\mathcal{E}(\varepsilon)$; this is valid for any balance $\deg \mathfrak{m}_\varepsilon : \deg \mathcal{E}(\varepsilon)$). Hence the size of this overdetermined algebraic system is further decreased.

Second, we use the unique admissible homogeneity weights for Kaup–Boussinesq system (1.4),

$$|u_0| = 1, \quad |u_{12}| = 2, \quad |\mathrm{d}/\mathrm{d}\xi| = 2;$$

here $|\mathrm{d}/\mathrm{d}x| \equiv 1$ is the normalization. The Miura contraction $\mathfrak{m}_\varepsilon = \big\{u_0 = \tilde{u}_0 + \varepsilon \cdot (\dots),$ $u_{12} = \tilde{u}_{12} + \varepsilon \cdot (\dots)\big\}$, which we assume regular at the origin, implies that $|\tilde{u}_0| = 1$ and $|\tilde{u}_{12}| = 2$ as well. We let $|\varepsilon| = -1$ be the difference of weights for every two successive Hamiltonians for the $N{=}2$, $a{=}4$–SKdV hierarchy, see [84] and (3.12) below. In this setup, all functional coefficients of the powers ε^k both in $\mathcal{E}(\varepsilon)$ and $\mathfrak{m}_\varepsilon$ are homogeneous differential polynomials in u_0, u_{12}, and their derivatives w.r.t. x. It is again important that the time ξ of weight $|\mathrm{d}/\mathrm{d}\xi| = 2$ in (3.8) precedes the time t with $|\mathrm{d}/\mathrm{d}t| = 3$ in the hierarchy of (1.2), where $|\theta_i| = -\frac{1}{2}$ and $|\boldsymbol{u}| = 1$. As before, we have further decreased the number of undetermined coefficients.

The polynomial ansatz for Gardner's deformation of (1.4) is generated by the procedure `GenSSPoly` with SsTools [73, 126]. We thus obtain the determining system $\mathfrak{m}_\varepsilon \colon \mathcal{E}(\varepsilon) \to \mathcal{E}$. Using SsTools and Crack, we split it to the overdetermined system of algebraic equations, which are linear w.r.t. $\mathcal{E}(\varepsilon)$ and quadratic-nonlinear w.r.t. $\mathfrak{m}_\varepsilon$. Moreover, we claim that this system is *triangular*. Indeed, it is ordered by the powers of ε, since the determining system is identically satisfied at zeroth order and because equations at lower orders of ε involve only the coefficients of its lower powers from $\mathfrak{m}_\varepsilon$ and $\mathcal{E}(\varepsilon)$.

Thirdly, we use the deformation (2.1) of the Korteweg–de Vries equation [102]. We recall that

- Miura's contraction $\mathfrak{m}_\varepsilon$ is common for all two-component systems of the Kaup–Boussinesq hierarchy;

[1]This estimate is rough and can be improved by operating separately with the components of $\mathfrak{m}_\varepsilon$ and $\mathcal{E}(\varepsilon)$ since, in particular, Kaup–Boussinesq system (1.4) is *linear* in u_{12}.

[2]Reductions other than (1.3) can produce quadratic-nonlinear subsystems of cubic-nonlinear system (1.2), *e.g.*, if one sets $u_0 = 0$ and $u_2 = 0$, see (3.13) on p. 38.

- for any a, the bosonic limit of (1.2), see (3.5) and (3.10), incorporates Korteweg–de Vries equation (1.1).

Using (2.1a), we fix those coefficients in $\mathfrak{m}_\varepsilon$ which depend only on u_{12} and its derivatives, but not on u_0 or its derivatives. Apparently, we discard the knowledge of such coefficients in the extension of bosonic limit (3.10), since for us now it is not the object to be deformed. But the minimization of the algebraic system, which we have achieved by passing to (3.8), is so significant that this temporary loss in inessential. Furthermore, the above reasoning shows that the recovery of the coefficients in the extension $\mathcal{E}(\varepsilon)$ amounts to solution of linear equations, while finding the coefficients in $\mathfrak{m}_\varepsilon$ would cost us the necessity to solve nonlinear algebraic systems. We managed to fix some of those constants for granted.

We finally remark that the normalization of at least one coefficient in the deformation problem cancels the redundant dilation of the parameter ε, which, otherwise, would remain until the end. This is our fourth simplification.[3]

We let the degrees $\deg \mathfrak{m}_\varepsilon = \deg \mathcal{E}(\varepsilon)$ be equal to four (c.f. [84]). Under this assumption, the two-component homogeneous polynomial extension $\mathcal{E}(\varepsilon)$ of system (1.4) contains 160 undetermined coefficients. At the same time, the two components of the Miura contraction $\mathfrak{m}_\varepsilon$ depend on 94 coefficients. However, we decrease this number by nine, setting the coefficient of $\tilde{u}_{12;x}$ equal to $+1$ and, similarly, to -1 for $\tilde{u}_{12}^2$ (see (2.1a), where the $\pm$ sign is absorbed by $\varepsilon \mapsto -\varepsilon$). Likewise, we set equal to zero the seven coefficients of $\tilde{u}_{12;xx}$, $\tilde{u}_{12}\tilde{u}_{12;x}$, $\tilde{u}_{12;xxx}$, $\tilde{u}_{12}^3$, $\tilde{u}_{12;x}^2$, $\tilde{u}_{12}\tilde{u}_{12;xx}$, and $\tilde{u}_{12;xxxx}$ in $\mathfrak{m}_\varepsilon$.

The resulting algebraic system with the shortened list of unknowns and with the auxiliary list of nine substitutions is handled by SsTools and then solved by using Crack [125].

Theorem 1 ([47])**.** Under the above assumptions, the Gardner deformation problem for Kaup–Boussinesq equation (1.4) has a unique real solution of degree 4. The Miura contraction $\mathfrak{m}_\varepsilon$ is given by

$$u_0 = \tilde{u}_0 + \varepsilon \tilde{u}_{0;x} - 2\varepsilon^2 \tilde{u}_{12}\tilde{u}_0, \tag{2.3a}$$

$$u_{12} = \tilde{u}_{12} + \varepsilon\big(\tilde{u}_{12;x} - 2\tilde{u}_0\tilde{u}_{0;x}\big) + \varepsilon^2\big(4\tilde{u}_{12}\tilde{u}_0^2 - \tilde{u}_{12}^2 - \tilde{u}_{0;x}^2\big) + 4\varepsilon^3\tilde{u}_{12}\tilde{u}_0\tilde{u}_{0;x} - 4\varepsilon^4\tilde{u}_{12}^2\tilde{u}_0^2. \tag{2.3b}$$

The extension $\mathcal{E}(\varepsilon)$ of (1.4) is

$$\tilde{u}_{0;\xi} = -\tilde{u}_{12;x} + 4u_0\tilde{u}_{0;x} + 2\varepsilon\big(\tilde{u}_0\tilde{u}_{0;x}\big)_x - 4\varepsilon^2\big(\tilde{u}_0^2 u_{12}\big)_x, \tag{2.4a}$$

$$\tilde{u}_{12;\xi} = \tilde{u}_{0;xxx} + 4\big(\tilde{u}_0\tilde{u}_{12}\big)_x - 2\varepsilon\big(\tilde{u}_0\tilde{u}_{12;x}\big)_x - 4\varepsilon^2\big(\tilde{u}_0\tilde{u}_{12}^2\big)_x. \tag{2.4b}$$

System (2.4) preserves the first Hamiltonian operator $\hat{A}_1^\varepsilon = \begin{pmatrix} 0 & \mathrm{d}/\mathrm{d}x \\ \mathrm{d}/\mathrm{d}x & 0 \end{pmatrix}$ from $\hat{A}_1^{12}$ for (1.4).

[3] There is one more possibility to reduce the size of the algebraic system: this can be achieved by a thorough balance of the *differential orders* of $\mathfrak{m}_\varepsilon$ and $\mathcal{E}(\varepsilon)$.

The Miura contraction $\mathfrak{m}_\varepsilon$ is shared by all equations in the Kaup–Boussinesq hierarchy. Solving the linear algebraic system, we find the extension $\big(\lim_B \mathcal{E}^{a=4}_{\text{SKdV}}\big)(\varepsilon)$ for the bosonic limit (3.10) of (1.2) with a=4:

$$\begin{aligned}
\tilde u_{0;t} = &-\tilde u_{0;xxx} - 6\big(\tilde u_0 \tilde u_{12}\big)_x + 12\tilde u_0^2 \tilde u_{0;x} + 12\varepsilon\big(\tilde u_0^2\tilde u_{0;x}\big)_x + 6\varepsilon^2\big(\tilde u_0\tilde u_{12}^2 - 4\tilde u_{12}\tilde u_0^3 + \tilde u_0 \tilde u_{0;x}^2)\big)_x \\
&+ \varepsilon^3\big((-24)\tilde u_{12}\tilde u_0^2\tilde u_{0;x}\big)_x + \varepsilon^4\big(24\tilde u_{12}^2\tilde u_0^3\big)_x, \qquad (2.5a)\\
\tilde u_{12;t} = &-\tilde u_{12;xxx} - 6\tilde u_{12}\tilde u_{12;x} + 12\big(\tilde u_0^2\tilde u_{12}\big)_x + 6\tilde u_0\tilde u_{0;xxx} + 12\tilde u_{0;xx}\tilde u_{0;x} \\
&+ 6\varepsilon\big(\tilde u_{0;xx}\tilde u_{0;x} - 2\tilde u_0^2\tilde u_{12;x}\big)_x \\
&+ 2\varepsilon^2\big(\tilde u_{12}^3 - 18\tilde u_{12}^2\tilde u_0^2 - 6\tilde u_{12}\tilde u_0\tilde u_{0;xx} - 3\tilde u_{12}\tilde u_{0;x}^2 - 6\tilde u_0\tilde u_{12;x}\tilde u_{0;x}\big)_x \\
&+ 24\varepsilon^3\big(\tilde u_{12}\tilde u_0^3\tilde u_{12;x}\big)_x + 24\varepsilon^4\big(\tilde u_{12}^3\tilde u_0^2\big)_x. \qquad (2.5b)
\end{aligned}$$

Now we expand the fields $\tilde u_0(\varepsilon) = \sum_{k=0}^{+\infty} \tilde u_0^{(k)} \cdot \varepsilon^k$ and $\tilde u_{12}(\varepsilon) = \sum_{k=0}^{+\infty} \tilde u_{12}^{(k)} \cdot \varepsilon^k$, and plug the formal power series for $\tilde u_0$ and $\tilde u_{12}$ in $\mathfrak{m}_\varepsilon$. Hence we start from $\tilde u_0^{(0)} = u_0$ and $\tilde u_{12}^{(0)} = u_{12}$, which is standard, and proceed with the recurrence relations between the conserved densities $u_0^{(k)}$ and $u_{12}^{(k)}$,

$$\begin{aligned}
&\tilde u_0^{(1)} = -u_{0;x}, \quad \tilde u_0^{(n)} = -\tfrac{\mathrm{d}}{\mathrm{d}x}\tilde u_0^{(n-1)} + \sum_{j+k=n-2} 2\tilde u_{12}^{(k)}\tilde u_0^{(j)}, \quad \forall n \geq 2;\\
&\tilde u_{12}^{(1)} = 2u_0u_{0;x} - u_{12;x}, \quad \tilde u_{12}^{(2)} = u_{12}^2 + u_{12;xx} - 4u_{12}u_0^2 - 3u_{0;x}^2 - 4u_0u_{0;xx},\\
&\tilde u_{12}^{(3)} = \sum_{j+k=2} 2\tilde u_0^{(j)}\tfrac{\mathrm{d}}{\mathrm{d}x}\tilde u_0^{(k)} - \tfrac{\mathrm{d}}{\mathrm{d}x}\tilde u_{12}^{(2)} + \sum_{j+k=1}\Big(\tilde u_{12}^{(j)}\tilde u_{12}^{(k)} + (\tfrac{\mathrm{d}}{\mathrm{d}x}\tilde u_0^{(j)})(\tfrac{\mathrm{d}}{\mathrm{d}x}\tilde u_0^{(k)})\Big)\\
&\qquad - \sum_{j+k+l=1} 4\tilde u_{12}^{(j)}\tilde u_0^{(k)}\tilde u_0^{(l)} - 4u_{12}u_0u_{0;x},\\
&\tilde u_{12}^{(n)} = -\tfrac{\mathrm{d}}{\mathrm{d}x}\tilde u_{12}^{(n-1)} + \sum_{j+k=n-1} 2\tilde u_0^{(j)}\tfrac{\mathrm{d}}{\mathrm{d}x}\tilde u_0^{(k)} + \sum_{j+k=n-2}(\tilde u_{12}^{(j)}\tilde u_{12}^{(k)} + (\tfrac{\mathrm{d}}{\mathrm{d}x}(\tilde u_0^{(j)})\tfrac{\mathrm{d}}{\mathrm{d}x}(\tilde u_0^{(k)}))\\
&\qquad - \sum_{j+k+l=n-2} 4\tilde u_{12}^{(j)}\tilde u_0^{(k)}\tilde u_0^{(l)} - \sum_{j+k+l=n-3} 4\tilde u_{12}^{(j)}\tilde u_0^{(k)}\tfrac{\mathrm{d}}{\mathrm{d}x}\tilde u_0^{(l)}\\
&\qquad + \sum_{j+k+l+m=n-4} 4\tilde u_{12}^{(j)}\tilde u_{12}^{(k)}\tilde u_0^{(l)}\tilde u_0^{(m)}, \qquad \forall n \geq 4.
\end{aligned}$$

Example 2. Following this recurrence, let us generate the eight lowest weight nontrivial conserved densities, which start the tower of Hamiltonians for the Kaup–Boussinesq hierarchy.

We begin with $\tilde u_0^{(0)} = u_0$ and $\tilde u_{12}^{(0)} = u_{12}$. Next, we obtain the densities

$$\tilde u_0^{(2)} = u_{0;xx} + 2u_0u_{12}, \qquad \tilde u_{12}^{(2)} = -4u_{0;xx}u_0 - 3u_{0;x}^2 + u_{12;xx} - 4u_0^2u_{12} + u_{12}^2,$$

which contribute to the tri-Hamiltonian representation of (1.4), see Proposition 1. Now we proceed with

$$\begin{aligned}
\tilde{u}_0^{(4)} &= u_{0;4x} - 12u_{0;xx}u_0^2 + 6u_{0;xx}u_{12} - 18u_{0;x}^2u_0 + 10u_{0;x}u_{12;x} + 6u_{12;xx}u_0 - 8u_0^3u_{12} + 6u_0u_{12}^2,\\
\tilde{u}_{12}^{(4)} &= -8u_{0;4x}u_0 - 20u_{0;xxx}u_{0;x} - 13u_{0;xx}^2 + 32u_{0;xx}u_0^3 - 48u_{0;xx}u_0u_{12} + 72u_{0;x}^2u_0^2 - 38u_{0;x}^2u_{12} -\\
&\quad - 80u_{0;x}u_{12;x}u_0 + u_{12;4x} - 24u_{12;xx}u_0^2 + 6u_{12;xx}u_{12} + 5u_{12;x}^2 + 16u_0^4u_{12} - 24u_0^2u_{12}^2 + 2u_{12}^3,\\
\tilde{u}_0^{(6)} &= u_{0;6x} - 40u_{0;4x}u_0^2 + 10u_{0;4x}u_{12} - 200u_{0;xxx}u_{0;x}u_0 + 28u_{0;xxx}u_{12;x} - 130u_{0;xx}^2u_0 -\\
&\quad - 198u_{0;xx}u_{0;x}^2 + 38u_{0;xx}u_{12;xx} + 80u_{0;xx}u_0^4 - 240u_{0;xx}u_0^2u_{12} + 30u_{0;xx}u_{12}^2 + 240u_{0;x}^2u_0^3 -\\
&\quad - 380u_{0;x}^2u_0u_{12} + 28u_{0;x}u_{12;xxx} - 400u_{0;x}u_{12;x}u_0^2 + 100u_{0;x}u_{12;x}u_{12} + 10u_{12;4x}u_0 -\\
&\quad - 80u_{12;xx}u_0^3 + 60u_{12;xx}u_0u_{12} + 50u_{12;x}^2u_0 + 32u_0^5u_{12} - 80u_0^3u_{12}^2 + 20u_0u_{12}^3,\\
\tilde{u}_{12}^{(6)} &= -12u_{0;6x}u_0 - 42u_{0;5x}u_{0;x} - 80u_{0;4x}u_{0;xx} + 160u_{0;4x}u_0^3 - 120u_{0;4x}u_0u_{12} - 49u_{0;xxx}^2 +\\
&\quad + 1200u_{0;xxx}u_{0;x}u_0^2 - 312u_{0;xxx}u_{0;x}u_{12} - 336u_{0;xxx}u_{12;x}u_0 + 780u_{0;xx}^2u_0^2 - 206u_{0;xx}^2u_{12} +\\
&\quad + 2376u_{0;xx}u_{0;x}^2u_0 - 716u_{0;xx}u_{0;x}u_{12;x} - 456u_{0;xx}u_{12;xx}u_0 - 192u_{0;xx}u_0^5 + 960u_{0;xx}u_0^3u_{12} -\\
&\quad - 360u_{0;xx}u_0u_{12}^2 + 297u_{0;x}^4 - 366u_{0;x}^2u_{12;xx} - 720u_{0;x}^2u_0^4 + 2280u_{0;x}^2u_0^2u_{12} - 290u_{0;x}^2u_{12}^2 -\\
&\quad - 336u_{0;x}u_{12;xxx}u_0 + 1600u_{0;x}u_{12;x}u_0^3 - 1200u_{0;x}u_{12;x}u_0u_{12} + u_{12;6x} - 60u_{12;4x}u_0^2 +\\
&\quad + 10u_{12;4x}u_{12} + 28u_{12;xxx}u_{12;x} + 19u_{12;xx}^2 + 240u_{12;xx}u_0^4 - 360u_{12;xx}u_0^2u_{12} + 30u_{12;xx}u_{12}^2 -\\
&\quad - 300u_{12;x}^2u_0^2 + 50u_{12;x}^2u_{12} - 64u_0^6u_{12} + 240u_0^4u_{12}^2 - 120u_0^2u_{12}^3 + 5u_{12}^4, \qquad \textit{etc.}
\end{aligned}$$

We will use these formulas in the next section, where, as an illustration, we re-derive the seven super-Hamiltonians of [84].

Theorem 2 ([47])**.** In the above notation, the following statements hold:

- The conserved densities $\tilde{u}_0^{(2k)}$ and $\tilde{u}_{12}^{(2k)}$ of weights $2k+1$ and $2k+2$, respectively, are nontrivial for all integers $k \geq 0$.

- Consider the zero-order components $\breve{u}_0(u_0, u_{12}, \varepsilon)$ and $\breve{u}_{12}(u_0, u_{12}, \varepsilon)$ of the series $\tilde{u}_0\big([u_0, u_{12}], \varepsilon\big)$ and $\tilde{u}_{12}\big([u_0, u_{12}], \varepsilon\big)$ with differential-polynomial coefficients. Then these generating functions are given by the formulas

$$\big(\breve{u}_0(u_0, u_{12}, \varepsilon^2)\big)^2 = \frac{1}{8\varepsilon^2} \cdot \left[4\varepsilon^2(u_0^2 + u_{12}) - 1 + \sqrt{1 + 8\varepsilon^2(u_0^2 - u_{12}) + 16\varepsilon^4(u_0^2 + u_{12})^2}\right], \tag{2.6a}$$

$$\breve{u}_{12}(u_0, u_{12}, \varepsilon^2) = \frac{1}{2\varepsilon^2} \cdot \left[1 - \sqrt{\tfrac{1}{2} - 2\varepsilon^2(u_{12} + u_0^2) + \tfrac{1}{2}\sqrt{1 + 8\varepsilon^2(u_0^2 - u_{12}) + 16\varepsilon^4(u_0^2 + u_{12})^2}}\right]. \tag{2.6b}$$

- The generating functions for the odd-index conserved densities $\tilde{u}_0^{(2k+1)}$ and $\tilde{u}_{12}^{(2k+1)}$ are expressed via the even-index densities, see (2.8) and (2.9), respectively. We claim that all the odd-index densities are trivial.

Proof. The densities $\tilde{u}_0^{(k)}$ and $\tilde{u}_{12}^{(k)}$, which are conserved for the bosonic limit (3.10) of the N=2, a=4–SKdV system (3.5), retract to the conserved densities for Korteweg–de Vries equation (1.1) under $u_0 \equiv 0$, see Example 1. The corresponding reduction of $\breve{u}_{12}(u_0, u_{12}, \varepsilon)$ is generating function (2.2). This implies that $\breve{u}_{12} = \sum_{k=0}^{+\infty} c(k) u_{12}^k \cdot \varepsilon^{2k} + \ldots$, whence the densities $\tilde{u}_{12}^{(2k)}$ are nontrivial.

Following the line of reasonings on p. 11, we consider the zero-order terms in Miura's contraction (2.3), which yields

$$u_0 = \breve{u}_0 \cdot \bigl(1 - 2\varepsilon^2 \breve{u}_{12}\bigr), \tag{2.7a}$$
$$u_{12} = \breve{u}_{12} + \varepsilon^2\bigl(4\breve{u}_0^2 \breve{u}_{12} - \breve{u}_{12}^2\bigr) - 4\varepsilon^4 \breve{u}_0^2 \breve{u}_{12}^2. \tag{2.7b}$$

Therefore,

$$\breve{u}_0 = \frac{u_0}{1 - 2\varepsilon^2 \breve{u}_{12}} = \sum_{k=0}^{+\infty} u_0 \cdot \bigl(2\varepsilon^2 \breve{u}_{12}\bigr)^k.$$

Since the coefficients $c(k)$ of $u_{12}^k \cdot \varepsilon^{2k}$ in $\breve{u}_{12}$ are positive, so are the coefficients of $u_0 u_{12}^k \cdot \varepsilon^{2k}$ in $\breve{u}_0$ for all $k \geq 0$. This proves that the conserved densities $\tilde{u}_0^{(2k)}$ are nontrivial as well.

Second, squaring (2.7a) and adding it to (2.7b), we obtain the equality $u_0^2 + u_{12} = \breve{u}_0^2 + \breve{u}_{12} - \varepsilon^2 \breve{u}_{12}^2$. In agreement with $\breve{u}_0\bigr|_{\varepsilon=0} = u_0$ and $\breve{u}_{12}\bigr|_{\varepsilon=0} = u_{12}$, we choose the root $\breve{u}_{12} = \bigl[1 - \sqrt{1 - 4\varepsilon^2 \cdot \bigl(u_{12} + u_0^2 - \breve{u}_0^2\bigr)}\bigr]/(2\varepsilon^2)$ of this quadratic equation. Hence (2.7a) yields the bi-quadratic equation upon $\breve{u}_0$,

$$1 - 4\varepsilon^2\bigl(u_{12} + u_0^2 - \breve{u}_0^2\bigr) = u_0^2/\breve{u}_0^2.$$

As above, the proper choice of its root gives (2.6a), whence we return to $\breve{u}_{12}$ and finally obtain (2.6b).

Finally, let us substitute the expansions $\tilde{u}_0 = v_0(\varepsilon^2) + \varepsilon \cdot v_1(\varepsilon^2)$ and $\tilde{u}_{12} = \omega_0(\varepsilon^2) + \varepsilon \cdot \omega_1(\varepsilon^2)$ in (2.3) for $\tilde{u}_0$ and $\tilde{u}_{12}$, see Example 1. By balancing the odd powers of ε in (2.3a), it is then easy to deduce the equality

$$v_1 \equiv \sum_{k=0}^{+\infty} \tilde{u}_0^{(2k+1)} \cdot \varepsilon^{2k} = \frac{1}{4\varepsilon^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \log\bigl(1 - 4\varepsilon^2 \cdot v_0\bigr), \qquad \text{where } v_0 \equiv \sum_{\ell=0}^{+\infty} \tilde{u}_0^{(2\ell)} \cdot \varepsilon^{2\ell}. \tag{2.8}$$

The balance of odd powers of ε in (2.3b) yields the algebraic equation upon ω_1, whence, in agreement with the initial condition $\omega_1(0) = \tilde{u}_{12}^{(1)}$, we choose its root

$$\begin{aligned}
\omega_1 = \Big[& 1 - 2\varepsilon^2\omega_0 + 4\varepsilon^2\upsilon_0^2 + 4\varepsilon^4\big(\upsilon_1^2 - 2\omega_0\upsilon_0^2 + \upsilon_0\upsilon_{1;x} + \upsilon_1\upsilon_{0;x}\big) - 8\varepsilon^6\upsilon_1^2\omega_0 \\
& - \Big(1 + 4\varepsilon^2\big(2\upsilon_0^2 - \omega_0\big) + 4\varepsilon^4\big(\omega_0^2 + 2\upsilon_0\upsilon_{1;x} - 8\omega_0\upsilon_0^2 + 2\upsilon_1\upsilon_{0;x} + 2\upsilon_1^2 + 4\upsilon_0^4\big) \\
& + 16\varepsilon^6\big(2\omega_0^2\upsilon_0^2 - 2\upsilon_1^2\omega_0 - \omega_0\upsilon_0\upsilon_{1;x} - \omega_0\upsilon_1\upsilon_{0;x} - 2\upsilon_0^2\upsilon_1\upsilon_{0;x} + 2\upsilon_1\upsilon_0\omega_{0;x} \\
& \qquad + 2\upsilon_1^2\upsilon_0^2 - 4\omega_0\upsilon_0^4 + 2\upsilon_0^3\upsilon_{1;x}\big) \\
& + 16\varepsilon^8\big(\upsilon_1^4 + 2\omega_0^2\upsilon_1^2 + 4\omega_0^2\upsilon_0^4 - 2\upsilon_1^2\upsilon_0\upsilon_{1;x} - 4\omega_0\upsilon_0^3\upsilon_{1;x} + 8\upsilon_1^2\omega_0\upsilon_0^2 + 2\upsilon_1^3\upsilon_{0;x} \\
& \qquad + \upsilon_0^2\upsilon_{1;x}^2 + \upsilon_1^2\upsilon_{0;x}^2 + 4\omega_0\upsilon_0^2\upsilon_1\upsilon_{0;x} - 2\upsilon_0\upsilon_{1;x}\upsilon_1\upsilon_{0;x}\big) \\
& + 64\varepsilon^{10}\big(\upsilon_0\upsilon_{1;x}\upsilon_1^2\omega_0 - 2\omega_0^2\upsilon_0^2\upsilon_1^2 - \upsilon_1^3\upsilon_{0;x}\omega_0 - \upsilon_1^4\omega_0\big) + 64\varepsilon^{12}\upsilon_1^4\omega_0^2\Big)^{1/2}\Big] \Big/ (16\varepsilon^6\upsilon_1\upsilon_0).
\end{aligned} \tag{2.9}$$

We claim that, using the balance of the even powers of ε in (2.3), the representation $\sum_{k=0}^{+\infty} \tilde{u}_{12}^{(2k+1)} \cdot \varepsilon^{2k} \in \operatorname{im} \frac{\mathrm{d}}{\mathrm{d}x}$ can be deduced, whence $\tilde{u}_{12}^{(2k+1)} \sim 0$. $\square$

2.4 Gardner's deformations as generators of new integrable systems

The aim of this section is to further and illustrate a practical concept which was outlined earlier by Kiselev in [59]. Namely, we revisit the problem of integrable deformation of a given infinite-dimensional system; the seminal paper was [102]. Much work towards description of the arising moduli spaces has been performed by Dubrovin *et al.* [32], cf. [34, 33]. (It must be recalled that cohomological theories in this context and organization of the moduli spaces are sensitive to the choice of admissible classes of differential functions – e. g., polynomial, rational, or analytic – in which such structures are sought for.) In the world of integrable systems there is a closely related aspect of integrability-preserving transition between (solutions to) systems of PDEs (e. g., via Bäcklund transformations, see [50]; a different approach was developed in [15]). Here one could employ the 'heavy artillery' [49, 50] of jet-bundle techniques for deformation of the Cartan structure elements in coverings over PDEs by using the Frölicher–Nijenhuis bracket, see Chapter 5 for illustration.

Having its roots in topological QFT and yet possessing numerous applications elsewhere, a task of extending low-order hierarchies with higher-order symbols remains a topic of particular interest in the field ([32], also [39, 40]). For instance, such is the approach to hydrodynamic-type systems viewed as the weak dispersion limits of larger, initially concealed models. Not limited to the above-mentioned class of evolution equations, this concept suits well for PDE systems of order $\geqslant 2$ whenever those are taken as drafts for the (re)construction of larger models; certain restrictions could be imposed by hand at exactly this moment in order to narrow, e. g., the classes of solutions of the draft systems, cf. [39, 40]. At the same time, there co-exist many schemes for extension of the symbol for a given system (e. g., one follows the perturbative approach of [32] or applies the Lax-pair based techniques from [82]).

From a broader perspective, there arise two natural questions: which systems are proclaimed 'interesting', thus delimiting the sets of start- and endpoints in the proliferation schemes, and where one could take those 'interesting' systems from — or pick the drafts of new interesting PDEs. Leaving now aside the ever-growing supply from Physics or a straightforward idea of ploughing the available lists of already known integrable systems, let us focus on a self-starting, regular algorithm which exploits the classical ideas from geometry of differential equations ([12, 62, 106]).

Specifically, we take the existence of infinitely many integrals of motion as a selection rule for nonlinear evolutionary systems; by default, we shall always assume that the collection of conversed quantities at hand is maximal, that is, it can not be extended within a class of conservation laws with local densities (otherwise, a count of infinities could become risky). As a rule, such systems tend to be bi-Hamiltonian at least in the case when the spatial dimension n is equal to 1, with $x^1 \equiv x$, see [91] and also [115]. Let us note also that a requirement of existence of conserved quantities is, generally speaking, stronger than a 'symmetry integrability' assumption [100]. (However in applications it is often convenient to weaken the former requirement in favour of the latter; we shall profit from the use of both approaches, see Example 4 and Proposition 2 in what follows.)

As soon as we agree to study only those evolutionary systems which admit infinite towers of integrals of motion, it is natural to first ex- and then inspect the existence of a (much better if polynomial) recurrence relation between the integrals' conserved densities [52, 59, 82, 102]. This yields a regular procedure for consecutive calculation of the integrals of motion on the basis of all previously known data by starting from the 'seed' constants. Let us emphasize that such relations between the densities are much more valuable and informative than ordinary recursion operators $R\colon \varphi_i \mapsto \varphi_{i+1}$ for symmetries or say, $R^\dagger\colon \psi_i \mapsto \psi_{i+1}$ for the 'cosymmetries' of evolutionary PDE; in a sense, every algorithm which explicitly produces the densities contains the built-in homotopy formula for reversion of the variational derivative that takes densities to the respective generating functions ψ, cf. [62, § 4.2] and [106].

The classical notion of Gardner's deformation $\mathfrak{m}_\varepsilon\colon \mathcal{E}(\varepsilon) \to \mathcal{E}$ for a completely integrable system $\mathcal{E}$ was designed for serving exactly this purpose [102]; in the course of years, it has become the parent structure for a plethora of concepts ranging from the Lax pair to formal τ-function, etc. (for a more general approach to the geometry of Gardner's deformations see [59]). Quite remarkably, this good old construction (see Definition 1) also answers the second question which we posed so far: for a system $\mathcal{E}$ which undergoes the deformation, this procedure yields "promising" drafts $\mathcal{E}'$ of "interesting" new systems $\mathcal{E}''$.

Kiselev sketched this line of reasoning in [59] and we now discuss it in more detail. Our surprising conclusion is that the world of completely integrable systems could be much more 'tense' and regularly organized than it may first seem; for the adjacency relations $\mathcal{E} \to \mathcal{E}''$ spin a web across that set, with topology still to be explored; to the best of our knowledge, a study of the physical sense for a property of two models $\mathcal{E}$ and $\mathcal{E}''$ to be adjacent has not yet begun.

Let us repeat the classical definition of Gardner's deformation for evolutionary PDE. Let $\mathcal{E} = \{u_t = f(x,[u])\}$ be a system of evolution equations (in particular, a completely integrable system). Suppose $\mathcal{E}(\varepsilon) = \{\tilde{u}_t = f_\varepsilon(x,[\tilde{u}],\varepsilon) \mid f_\varepsilon \in \operatorname{im}\frac{\mathrm{d}}{\mathrm{d}x}\}$ is a deformation of $\mathcal{E}$ such that at each point $\varepsilon \in \mathcal{I}$ of an interval $\mathcal{I} \subseteq \mathbb{R}$ there is the *Miura contraction* $\mathfrak{m}_\varepsilon = \{u = u([\tilde{u}],\varepsilon)\} \colon \mathcal{E}(\varepsilon) \to \mathcal{E}$. Then the pair $(\mathcal{E}(\varepsilon),\mathfrak{m}_\varepsilon)$ is the (*classical*) *Gardner deformation* for system $\mathcal{E}$.

We say that the coefficient $\varphi(\tilde{u},\tilde{u}_x,\dots)$ of the highest power of ε in the right-hand side of a *polynomial* (in ε) Gardner's extension $\mathcal{E}(\varepsilon)$ determines the *adjoint* system $\mathcal{E}' = \{\tilde{u}_{t_k} = \varphi\}$.

Example 3. The evolution equation

$$\tilde{u}_{t_3} = -6\tilde{u}^2\tilde{u}_x \qquad (t_3 = t \text{ in } (1.1)) \tag{2.10}$$

is adjoint to Korteweg–de Vries equation (1.1) with respect to its Gardner's deformation (2.1).

We notice that the adjoint systems are often dispersionless, although this is not always the case (cf. [59] and [52]). Let us now address the natural problem of extension of the new, adjoint equation – and its hierarchy which appears by construction of Gardner's deformation – by adding terms with higher-order derivatives (in particular, by switching on the dispersion in $\mathcal{E}'$). There are many techniques for solving this problem: a straightforward computational algorithm, which does not require that the adjoint system be hydrodynamic type, is illustrated in what follows by using the 'symmetry integrability' approach (2.10) and software [73]; we then report on the second iteration of such proliferation scheme and discuss the third and other steps to follow. We use capital letters for unknowns in extensions of adjoint system.

Example 4. It is readily seen that equation (2.10) is the second element in the infinite hierarchy of adjoint systems (corresponding to Gardner's extensions of higher KdV flows, with the Miura contraction $\mathfrak{m}_\varepsilon$ common for all of them), which is

$$\begin{aligned}
&\vdots && \vdots\\
&\mathcal{E}'_{\mathrm{KdV}_2} = \{\tilde{u}_{12;t_5} = \tilde{u}_{12}^4\tilde{u}_{12;x}\}, && \varphi_5 = \tilde{u}_{12}^4\tilde{u}_{12;x},\\
&\mathcal{E}'_{\mathrm{KdV}_1} = \{\tilde{u}_{12;t_3} = -6\tilde{u}_{12}^2\tilde{u}_{12;x}\}, && \varphi_3 = -6\tilde{u}_{12}^2\tilde{u}_{12;x},\\
&\mathcal{E}'_{\mathrm{KdV}_0} = \{\tilde{u}_{12;t_1} = \tilde{u}_{12;x}\}, && \varphi_1 = \tilde{u}_{12;x}.
\end{aligned}$$

Clearly, the scaling weights are not uniquely determined for the dependent variables in the adjoint hierarchy; for definition, set $[U]=1$, $[\mathrm{d}/\mathrm{d}x]=1$, and $[\mathrm{d}/\mathrm{d}t_k]=(k-1)[U]+[\mathrm{d}/\mathrm{d}x]=k$, which is consistent with the dynamics. For all $k\in\mathbb{N}$, let us now list all scaling-homogeneous differential polynomials f_k of weights $[f_k]=[U]+[\mathrm{d}/\mathrm{d}t_k]=k+1$ with undetermined coefficients, excluding at once those terms which are already contained in the respective right-hand sides of the adjoint hierarchy $\mathcal{E}'_{\mathrm{KdV}_k}$. For instance, we let

$$\begin{aligned}
&\vdots\\
f_5 &= q_5U^6+q_6U^3U_{xx}+q_7U^2U_{xxx}+q_8U^2(U_x)^2+q_9UU_{4x}+q_{10}UU_{xx}U_x+q_{11}U_{5x}+q_{12}U_{xxx}U_x\\
&\quad+q_{13}(U_{xx})^2+q_{14}(U_x)^3,\\
f_3 &= q_1U^4+q_2UU_{xx}+q_3U_{xxx}+q_4(U_x)^2,\\
f_1 &= 0.
\end{aligned}$$

The Ansatz for the full hierarchy is thus

$$\begin{aligned}
U_{t_k} &= \varphi_k+f_k,\\
&\quad\dots\\
U_{t_5} &= \varphi_5+f_5=U^4U_x+q_5U^6+q_6U^3U_{xx}+q_7U^2U_{xxx}+q_8U^2(U_x)^2+q_9UU_{4x}+q_{10}UU_{xx}U_x\\
&\quad+q_{11}U_{5x}+q_{12}U_{xxx}U_x+q_{13}(U_{xx})^2+q_{14}(U_x)^3,\\
U_{t_3} &= \varphi_3+f_3=-6U^2U_x+q_1U^4+q_2UU_{xx}+q_3U_{xxx}+q_4(U_x)^2,\\
U_{t_1} &= \varphi_1+f_1=U_x.
\end{aligned}$$

By solving the determining system of algebraic equations $(U_{t_i})_{t_j}=(U_{t_j})_{t_i}$ upon the undetermined coefficients q_α and then taking its nontrivial solution (if any), we obtain a new, dispersionful hierarchy (which is symmetry integrable by construction). Specifically, for (1.1) and its adjoint (2.10) the solution is

$$\begin{aligned}
&\vdots\\
U_{t_5} &= \tfrac{1}{30}\bigl(U_{4x}+6U^5+10U^2U_{xx}+10U(U_x)^2\bigr)_x\\
U_{t_3} &= -U_{xxx}-6U^2U_x,\\
U_{t_1} &= U_x,
\end{aligned}$$

which is none other than the hierarchy of modified Korteweg–de Vries equation.

Consider the second term in one of the two towers of Kaup–Boussinesq hierarchy (1.4). Its Gardner deformation (2.3) is known from Section 2.3; let us recall that the extended equations are (here $\xi=t_2$)

$$\tilde{u}_{0;t_2}=-\tilde{u}_{12;x}+4u_0\tilde{u}_{0;x}+2\varepsilon\bigl(\tilde{u}_0\tilde{u}_{0;x}\bigr)_x-4\varepsilon^2\bigl(\tilde{u}_0^2u_{12}\bigr)_x,\tag{2.4a}$$
$$\tilde{u}_{12;t_2}=\tilde{u}_{0;xxx}+4\bigl(\tilde{u}_0\tilde{u}_{12}\bigr)_x-2\varepsilon\bigl(\tilde{u}_0\tilde{u}_{12;x}\bigr)_x-4\varepsilon^2\bigl(\tilde{u}_0\tilde{u}_{12}^2\bigr)_x.\tag{2.4b}$$

By definition, the adjoint system is

$$\tilde{u}_{0;t_k} = (\tilde{u}_0^k \tilde{u}_{12}^{k-1})_x, \qquad \tilde{u}_{12;t_k} = (\tilde{u}_0^{k-1} \tilde{u}_{12}^k)_x. \tag{2.12}$$

Proposition 2 ([67])**.** Let us require that the dispersionful extension of (2.12) itself is an infinite-dimensional integrable system and that it is scaling-invariant with respect to the weights $[U_0] = [U_{12}] = \frac{1}{2}$ and $[\mathrm{d}/\mathrm{d}t_k] = k$ for $k \in \mathbb{N}$. Then there is a unique solution to the extension problem:

$$\begin{aligned} &\vdots && \vdots \\ U_{0;t_3} &= (U_{0;xx} + 6U_0 U_{0;x} U_{12} + 6U_0^3 U_{12}^2)_x, & U_{12;t_3} &= (U_{12;xx} - 6U_0 U_{12} U_{12;x} + 6U_0^2 U_{12}^3)_x, \\ U_{0;t_2} &= U_{0;xx} + 2(U_0^2 U_{12})_x, & U_{12;t_2} &= -U_{12;xx} + 2(U_0 U_{12}^2)_x, \\ U_{0;t_1} &= U_{0;x}, & U_{12;t_1} &= U_{12;x}. \end{aligned}$$

This is the Kaup–Newell hierarchy [55].

It would be quite logical to iterate the reasoning by first constructing a Gardner's deformation – or several such deformations – for the Kaup–Newell system, and then by extending the available adjoint system(s). However, this algorithmically simple problem appears unexpectedly complex as far as computations are concerned. Specifically, by using the analytic software [73] we obtain a 'no-go' result: there is no Gardner's deformation for the Kaup–Newell equation under the following set of assumptions:

- we supposed that the deformation $(\mathcal{E}(\varepsilon), \mathfrak{m}_\varepsilon)$ is polynomial in ε and differential polynomial in $\tilde{U}_0$ and $\tilde{U}_{12}$;
- we let such deformations be scaling homogeneous with respect to the weights $[\tilde{U}_0] = [\tilde{U}_{12}] = \frac{1}{2}$ and $[\varepsilon] = -\frac{1}{2}$;
- the polynomial Ansätze for Gardner's deformations were bounded by using $\deg_\varepsilon(\mathfrak{m}_\varepsilon) \leqslant 5$ and $\deg_\varepsilon(\mathcal{E}(\varepsilon)) \leqslant 10$ (here we note that $\max(\deg_\varepsilon \mathfrak{m}_\varepsilon) = 2 \times \max(\deg_\varepsilon(\mathcal{E}(\varepsilon)))$ for the Kaup–Newell system).

Let as also note that the extended equation $\mathcal{E}(\varepsilon)$ for the Kaup–Newell system can depend on derivatives of $\tilde{U}_0$ and $\tilde{U}_{12}$ with respect to x of orders up to but not exceeding *two*.[4] We expect that the Kaup–Newell system can be Gardner deformed strictly outside the class of differential polynomials (but can not be deformed within such class of functions).

[4]The proof is as follows: Consider the determining equation $(\mathfrak{m}_\varepsilon)_t = f(\mathfrak{m}_\varepsilon)$ for Gardner's deformation and calculate the differential orders of both sides; by the chain rule, this yields that $\mathrm{ord}_x(\mathfrak{m}_\varepsilon) + \mathrm{ord}_x(f_\varepsilon) = \mathrm{ord}_x(f) + \mathrm{ord}_x(\mathfrak{m}_\varepsilon)$, which implies a rough estimate $\mathrm{ord}_x(f_\varepsilon) = \mathrm{ord}_x(f)$.

It therefore remains an open problem to find Gardner's deformation(s) for the Kaup–Newell system and extend the arising adjoint equation(s) so that new, higher-order completely integrable hierarchies are attained.

We conclude that Gardner's deformations of infinite-dimensional completely integrable systems can be effectively used not only through their '$\mathfrak{m}_\varepsilon$-parts,' which encode the recurrence relations between conserved densities, but – viewed via their '$\mathcal{E}(\varepsilon)$-parts' – as a source of new completely integrable systems, or draft approximations to larger systems for which the integrability is retained.

The reproduction process is self-starting. Moreover, whenever there is a Gardner deformation for the new hierarchy, one could attempt another iteration. This scheme yields the oriented graph whose vertices are integrable systems and whose edges associate new such systems to the ones at their starting points. We emphasize that the degree of a vertex can be greater than two, meaning that a given system admits several deformations (cf. [52, 59]), and that, in principle, multiple edges may occur. A study of topology of such graph and its correlation with the structure of moduli spaces for higher perturbations of low-order models is a challenging open problem.

Chapter 3

Gardner's deformations of $\mathbb{Z}_2$-graded equations

3.1 Preliminaries: the $\mathbb{Z}_2$-graded infinite jet bundles

In this section we recall necessary definitions from supergeometry (we refer to [10, 30, 87] and [12, 42, 62, 78, 106] for further detail); this material is standard.

Let M^n be an n-dimensional smooth manifold. Let us consider two vector bundles over the same base M^n, $\pi^0\colon E_{\bar 0}^{m_0+n} \to M^n$ and $\pi^1\colon E_{\bar 1}^{m_1+n} \to M^n$ with fibre dimensions m_0 and m_1, respectively.[1] Let $\pi^{\bar 1} = \Pi\pi^1$ be the odd neighbour of the vector bundle π^1. By definition, this neighbour is the vector bundle $\pi^{\bar 1}\colon \Pi E_{\bar 1}^{n+m_1} \to M^n$ over the same base and with the same vector space $\mathbb{R}^{m_1}$ take as prototype for the fibers. The coordinates $\xi^1, \ldots, \xi^{m_1}$ along the fibers $(\pi^{\bar 1})^{-1}(x) \simeq \mathbb{R}^{m_1}$ are proclaimed $\mathbb{Z}_2$-parity odd, i.e., we introduce the $\mathbb{Z}_2$-grading $\mathsf{p}\colon x^i \mapsto \bar 0$, $\xi^k \mapsto \bar 1$ for the ring of smooth $\mathbb{R}$-valued functions on the total space $\Pi E_{\bar 1}^{m_1+n}$ of the superbundle; the grading then acts by multiplicative (semi)group homomorphism $\mathsf{p}\colon C^\infty(\Pi E_{\bar 1}^{m_1+n}) \to \mathbb{Z}_2 = \bigl(\{1,-1\},\times\bigr) \simeq \bigl(\{\bar 0, \bar 1\},+\bigr)$. We have that $C^\infty(\Pi E_{\bar 1}^{m_1+n}) \simeq \Gamma(\bigwedge^\bullet (E_{\bar 1}^{m_1+n})^*)$, where $(E_{\bar 1}^{m_1+n})^*$ denotes the space of fibrewise-linear functions on Σ^{m_1+n}. By construction, the new space of graded coordinate functions on $\Pi E_{\bar 1}^{m_1+n}$ is an $\mathbb{R}$-algebra and a $C^\infty(M^n)$-module. Finally, let us construct the Whitney sum $\pi = \pi^{\bar 0} \times_{M^n} \pi^{\bar 1}$ of the bundles $\pi^{\bar 0} = \pi^0$ and $\pi^{\bar 1}$ over the base M^n.

Consider the jet space $J^\infty(\pi)$ of sections of the superbundle π. Namely, for the superbundle π we define the infinite jet superbundle $\pi_\infty\colon J^\infty(\pi) \to M^n$ as follows: we let $(\pi_\infty)^{\bar 0} = (\pi^{\bar 0})_\infty$, $(\pi_\infty)^{\bar 1} = \Pi((\pi^1)_\infty)$ (see [56] for details). The set of variables describing $J^\infty(\pi)$ is composed by

- even coordinates x^i on M^n,
- even coordinates u^j and parity-odd coordinates ξ^k along the fibres of π; these objects themselves are elements of the set of

[1] In particular, we let $n = 2$ so that the independent variables are $x^1 = x$ and $x^2 = t$; we have that $m_0 = 1$, $m_1 = 0$ for the Korteweg–de Vries equation, $m_0 = 2$, $m_1 = 0$ for the hierarchy of the Kaup–Boussinesq equation, and $m_0 = 2$, $m_1 = 2$ for the N=2 supersymmetric KdV equation, see [10, 84, 99].

- even variables u^j_σ and parity-odd variables ξ^k_σ for the fibres of the infinite jet bundle $\pi_\infty\colon J^\infty(\pi) \to M^n$.

In the above notation we let σ be the multi-index that labels partial derivatives of the unknowns u^j and ξ^k w.r.t. even variables x^i; by convention, $u^j_\varnothing \equiv u^j$ and $\xi^k_\varnothing \equiv \xi^k$. The parity function p on homogeneous elements of $C^\infty(J^\infty(\pi))$ by its acting on generators

$$
\begin{aligned}
&\mathsf{p}(x^i) = \bar{0}, \qquad && \mathsf{p}(u^j) = \bar{0}, \qquad && \mathsf{p}(\xi^k) = \bar{1}, && \\
& && \mathsf{p}(u^j_\sigma) = \bar{0}, \qquad && \mathsf{p}(\xi^k_\sigma) = \bar{1}, \qquad && |\sigma| > 0,
\end{aligned}
$$

and satisfying following rules

$$
\begin{aligned}
\mathsf{p}(a \cdot b) &= \mathsf{p}(a) + \mathsf{p}(b),\\
\mathsf{p}(a + b) &= \mathsf{p}(a) = \mathsf{p}(b), \quad \text{iff } \mathsf{p}(a) = \mathsf{p}(b),
\end{aligned}
$$

where $a, b \in C^\infty(J^\infty(\pi))$.

The left *total derivatives* on $J^\infty(\pi)$ are expressed by the formula

$$
D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^{m_0} \sum_{\sigma_{\bar{0}}} u^j_{\sigma_{\bar{0}}+1_i} \frac{\partial}{\partial u^j_{\sigma_{\bar{0}}}} + \sum_{k=1}^{m_1} \sum_{\sigma_{\bar{1}}} \xi^k_{\sigma_{\bar{1}}+1_i} \frac{\vec{\partial}}{\partial \xi^k_{\sigma_{\bar{1}}}}.
$$

These vector fields commute (in a usual sense, even though the objects D_{x^i} contain the directed derivations). By definition, we put $D_\tau = D^{\tau_1}_{x^1} \circ \cdots \circ D^{\tau_n}_{x^n}$.

Let us recall the definition of a system of partial differential equations and its prolongation in context of $\mathbb{Z}_2$-graded setup. Consider a system of partial differential equations

$$
\mathcal{E} = \left\{ F^\ell(x^i, u^j, \ldots, u^j_{\sigma_{\bar{0}}}, \xi^k, \ldots, \xi^k_{\sigma_{\bar{1}}}) = 0, \qquad \ell = 1, \ldots, r \right\};
$$

without any loss of generality for applications we assume that the system at hand satisfies some mild assumptions which are outlined in [12, 62, 106]. Then the system $\mathcal{E}$ and all its differential consequences $D_\sigma(F^\ell) = 0$ (thus presumed existing, regular, and not leading to any contradiction in the course of derivation) generate the infinite prolongation $\mathcal{E}^\infty$ of the system $\mathcal{E}$.

Like in non-graded case for the $\mathbb{Z}_2$-graded setup we have that the de Rham differential $\bar{\mathrm{d}}$ on $\mathcal{E}^\infty$ is subjected to the decomposition $\bar{\mathrm{d}} = \bar{\mathrm{d}}_h + \bar{\mathrm{d}}_{\mathcal{C}}$, where $\bar{\mathrm{d}}_h\colon \Lambda^{p,q}(\mathcal{E}^\infty) \to \Lambda^{p,q+1}(\mathcal{E}^\infty)$ is the horizontal differential and $\bar{\mathrm{d}}_{\mathcal{C}}\colon \Lambda^{p,q}(\mathcal{E}^\infty) \to \Lambda^{p+1,q}(\mathcal{E}^\infty)$ is the vertical differential.

The differential $\bar{\mathrm{d}}_h$ can be expressed in coordinates by inspection of its action on elements of $C^\infty(\mathcal{E}^\infty) = \Lambda^{0,0}(\mathcal{E}^\infty)$, whence for any ϕ we have that

$$
\bar{\mathrm{d}}_h \phi = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \bar{D}_{x^i}(\phi), \tag{3.1a}
$$

$$
\bar{\mathrm{d}}_{\mathcal{C}} \phi = \sum_{j=1}^{m_0} \sum_{\sigma_{\bar{0}}} \omega^j_{\sigma_{\bar{0}}} \wedge \frac{\partial \phi}{\partial u^j_{\sigma_{\bar{0}}}} + \sum_{k=1}^{m_1} \sum_{\sigma_{\bar{1}}} \zeta^k_{\sigma_{\bar{1}}} \wedge \frac{\vec{\partial} \phi}{\partial \xi^j_{\sigma_{\bar{1}}}}, \tag{3.1b}
$$

where we put

$$\omega^j_{\sigma_{\bar 0}} = \mathrm{d}u^j_{\sigma_{\bar 0}} - \sum_{j=1}^{n} u^j_{\sigma_{\bar 0}+1_i}\,\mathrm{d}x^i, \qquad\qquad \zeta^j_{\sigma_{\bar 1}} = \mathrm{d}\xi^j_{\sigma_{\bar 1}} - \sum_{i=1}^{n} \xi^j_{\sigma_{\bar 1}+1_i}\,\mathrm{d}x^i.$$

We note further that $\mathrm{d}x^i$, $\mathrm{d}u^j_{\sigma_{\bar 0}}$, and $\mathrm{d}\xi^k_{\sigma_{\bar 1}}$ satisfy the following commutation relations:

$$\begin{aligned} \mathrm{d}x^i \wedge \mathrm{d}x^j &= -\mathrm{d}x^j \wedge \mathrm{d}x^i, & \mathrm{d}x^i \wedge \mathrm{d}u^j_{\sigma_{\bar 0}} &= -\mathrm{d}u^j_{\sigma_{\bar 0}} \wedge \mathrm{d}x^i, & \mathrm{d}x^i \wedge \mathrm{d}\xi^k_{\sigma_{\bar 1}} &= -\mathrm{d}\xi^k_{\sigma_{\bar 1}} \wedge \mathrm{d}x^i, \\ \mathrm{d}u^j_{\sigma_{\bar 0}} \wedge \mathrm{d}u^k_{\tau_{\bar 0}} &= -\mathrm{d}u^k_{\tau_{\bar 0}} \wedge \mathrm{d}u^j_{\sigma_{\bar 0}}, & \mathrm{d}\xi^k_{\sigma_{\bar 1}} \wedge \mathrm{d}u^j_{\tau_{\bar 0}} &= -\mathrm{d}u^j_{\tau_{\bar 0}} \wedge \mathrm{d}\xi^k_{\sigma_{\bar 1}}, & \mathrm{d}\xi^k_{\sigma_{\bar 1}} \wedge \mathrm{d}\xi^j_{\tau_{\bar 1}} &= +\mathrm{d}\xi^j_{\tau_{\bar 1}} \wedge \mathrm{d}\xi^k_{\sigma_{\bar 1}}; \end{aligned}$$

we refer to [63] for the geometric theory of variations in the frames of which one discovers why differential one-forms should anticommute in the $\mathbb{Z}_2$-graded sense.

The *substitution* of a $\mathbb{Z}_2$-graded vector field X into a $\mathbb{Z}_2$-graded differential form ω is defined by the formula $\mathrm{i}_X(\omega) = (-1)^{\mathsf{p}(X)\cdot\mathsf{p}(\omega)}\omega(X)$. We have that

$$\mathrm{i}_{\bar D_{x^i}}(\omega^j_{\sigma_{\bar 0}}) = \mathrm{i}_{\bar D_{x^i}}(\zeta^k_{\sigma_{\bar 1}}) = 0 \qquad \text{for all } i, j, k \text{ and } |\sigma| \geqslant 0.$$

These equalities mean that the Cartan distribution can be described equivalently in terms of the Cartan forms $\omega^j_{\sigma_{\bar 0}}$ and $\zeta^k_{\sigma_{\bar 1}}$.

The restriction of Cartan's distribution from $J^\infty(\pi)$ onto $\mathcal{E}^\infty$ is horizontal with respect to the projection $\pi_\infty\big|_{\mathcal{E}^\infty} \colon \mathcal{E}^\infty \to M$. This determines the connection $\mathcal{C}_{\mathcal{E}^\infty} \colon \mathrm{D}(M) \to \mathrm{D}(\mathcal{E}^\infty)$, where $\mathrm{D}(M)$ and $\mathrm{D}(\mathcal{E}^\infty)$ are the $C^\infty(M)$- and $C^\infty(\mathcal{E}^\infty)$-modules of vector fields on M and $\mathcal{E}^\infty$, respectively. We denote by $\mathrm{D}(\Lambda^1(\mathcal{E}^\infty))$ the $C^\infty(\mathcal{E}^\infty)$-module of derivations $C^\infty(\mathcal{E}^\infty) \to \Lambda^1(\mathcal{E}^\infty)$ taking values in the $C^\infty(\mathcal{E}^\infty)$-module of one-forms on $\mathcal{E}^\infty$. The connection form $U_{\mathcal{E}^\infty} \in \mathrm{D}(\Lambda^1(\mathcal{E}^\infty))$ of $\mathcal{C}_{\mathcal{E}^\infty}$ is called the *structural element* of the equation $\mathcal{E}^\infty$.

Let q be a natural number. Consider a superbundle π with fibre dimensions $m_{\bar 0} = 2^{q-1}$ and $m_{\bar 1} = 2^{q-1}$. Let $\mathfrak{G}_q$ be a Grassmann algebra with q odd generators $\theta_1, \ldots, \theta_q$. The tensor product $\pi_{N=q} = \mathfrak{G}_q \otimes_{\mathbb{R}} \pi$ of the Grassmann algebra $\mathfrak{G}_q$ and superbundle π is called the $N{=}q$ superbundle. The $N{=}q$ superfield $\boldsymbol{u}$ is a section $\boldsymbol{s} \in \Gamma(\pi_{N=q})$ of the $N{=}q$ superbundle $\pi_{N=q}$. We extend the definition of parity function p to $C^\infty(J^\infty(\pi_{N=q})) = \mathfrak{G}_q \otimes_{\mathbb{C}} C^\infty(J^\infty(\pi))$ by the formula $\mathsf{p}(\boldsymbol{u}) = \mathsf{p}(\mathsf{q}_i \otimes f^i) = \mathsf{p}(q_i) + \mathsf{p}(f^i)$, where $\mathsf{q}_i \in \mathfrak{G}_q$ and $f^i \in C^\infty(J^\infty(\pi))$. Let us consider only parity-homogeneous superfields. The superfield $\boldsymbol{u}$ is called bosonic if $p(\boldsymbol{u}) = \bar 0$. The superfield $\boldsymbol{u}$ is called fermionic if $\mathsf{p}(\boldsymbol{u}) = \bar 1$. The function f^i is called a bosonic component of the superfield $\boldsymbol{u}$ if $\mathsf{p}(f^i) = \bar 0$ and f^i is called a fermionic component of the superfield $\boldsymbol{u}$ if $\mathsf{p}(f^i) = \bar 1$. For example, for an $N{=}2$ superfield $\boldsymbol{u}$ we have that

$$\boldsymbol{u} = 1 \otimes u_0 + \theta_1 \otimes u_1 + \theta_2 \otimes u_2 + \theta_1\theta_2 \otimes u_{12}, \tag{3.2}$$

where θ_1 and θ_2 are Grassmann variables satisfying $\theta_1^2 = \theta_2^2 = \theta_1\theta_2 + \theta_2\theta_1 = 0$; and u_0, u_1, u_2, u_{12} are the fibre coordinates in π. In what follows we will omit the tensor product sign.

3.2 N=2 supersymmetric Korteweg–de Vries equations

Let us consider the N=2 supersymmetric Korteweg–de Vries equation [84, 85],

$$\boldsymbol{u}_t = -\boldsymbol{u}_{xxx} + 3\big(\boldsymbol{u}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}\big)_x + \frac{a-1}{2}\big(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}^2\big)_x + 3a\boldsymbol{u}^2\boldsymbol{u}_x, \qquad \mathcal{D}_i = \frac{\vec{\partial}}{\partial\theta_i} + \theta_i\cdot\frac{\mathrm{d}}{\mathrm{d}x}, \quad (1.2)$$

upon a scalar, complex bosonic N=2 superfield $\boldsymbol{u}$. For a=4, this super-equation possesses an infinite hierarchy of bosonic Hamiltonian super-functionals $\boldsymbol{\mathcal{H}}^{(k)}$ whose densities $\boldsymbol{h}^{(k)}$ are integrals of motion. We study whether these super-Hamiltonians can be produced recursively by using those which are already obtained. In particular, this can be done via Gardner's deformations [99, 102], which suggests finding a parametric family of super-equations $\mathcal{E}(\varepsilon)$ upon the generating super-function $\tilde{\boldsymbol{u}}(\varepsilon) = \sum_{k=0}^{+\infty}\boldsymbol{h}^{(k)}\cdot\varepsilon^k$ for the integrals of motion such that initial super-equation (1.2) is $\mathcal{E}(0)$. It is further supposed that, at each ε, the evolutionary equation $\mathcal{E}(\varepsilon)$ is given in the form of a (super-)conserved current, and there is the Gardner–Miura substitution $\mathfrak{m}_\varepsilon\colon \mathcal{E}(\varepsilon)\to\mathcal{E}(0)$. Hence, expanding $\mathfrak{m}_\varepsilon$ in ε and using the initial condition $\tilde{\boldsymbol{u}}(0) = \boldsymbol{u}$ at $\varepsilon = 0$, one obtains the differential recurrence relation between the Taylor coefficients $\boldsymbol{h}^{(k)}$ of the generating function $\tilde{\boldsymbol{u}}$ (see [102] or [4, 52, 59, 82, 84] and references therein for details and examples).

Let us summarize our main result. Under some natural assumptions, we prove the absence of N=2 supersymmetry-invariant Gardner's deformations for the bi-Hamiltonian N=2, a=4–SKdV. Still, we show that the deformation problem must be addressed in a different way, and then we solve it in two steps. First, in section 3.3 we recall that the tri-Hamiltonian hierarchy for the bosonic limit of (1.2) with a=4 contains the Kaup–Boussinesq equation, see [18, 54, 105] and [19, 83, 108]. In section 2.3 we construct new deformations for the Kaup–Boussinesq equation such that the Miura contraction $\mathfrak{m}_\varepsilon$ incorporates Gardner's map for the KdV equation ([102], c.f. [52, 82]). Second, extending the Hamiltonians $H^{(k)}$ for the Kaup–Boussinesq hierarchy to the super-functionals $\boldsymbol{\mathcal{H}}^{(k)}$ in section 3.5, we reproduce the bosonic conservation laws for (1.2) with a=4. Finally, we describe necessary conditions upon a class of Gardner's deformations for (1.2) that reproduce its *fermionic* local conserved densities (c.f. [99]).

Remark 1. The recurrence relations between the (super-)Hamiltonians of the hierarchy are much more informative than the usual recursion operators that propagate symmetries. In particular, the symmetries can be used to produce new explicit solutions from known ones, but the integrals of motion help to find those primary solutions.

Let us also note that, within the Lax framework of super-pseudodifferential operators, calculation of the $(n+1)$-th residue does not take into account the n residues which are already known at smaller indices. This is why the method of Gardner's deformations becomes highly preferrable. Indeed, there is no need to multiply any pseudodifferential operators by applying the Leibniz rule an increasing number of times, and all the previously obtained quantities are used at each inductive step. By this argument, we understand Gardner's deformations as the transformation in the space of the integrals of motion that maps the residues to Taylor coefficients of the generating functions $\tilde{\boldsymbol{u}}(\varepsilon)$ and which, therefore, endows this space with additional structure (that is, with the recurrence relations between the integrals).

Still there is a deep intrinsic relation between the Lax (or, more generally, zero-curvature) representations for integrable systems and Gardner's deformations for them. Namely, both approaches manifest the matrix and vector field representations of Lie algebras related to such systems, and the deformation parameter ε is inverse proportional to the eigenvalue in the linear spectral problem [123] (see Chapter 5 for details).

3.3 N=2 a=4–SKdV as bi-Hamiltonian super-extension of Kaup–Boussinesq system

Let us begin with the Korteweg–de Vries equation (1.1). Its second Hamiltonian operator, $\hat{A}_2^{\mathrm{KdV}} = \mathrm{d}^3/\mathrm{d}x^3 + 4u_{12}\,\mathrm{d}/\mathrm{d}x + 2u_{12;x}$, which relates (1.1) to the functional $H^{(2)}_{\mathrm{KdV}} = -\frac{1}{2}\int u_{12}^2\,\mathrm{d}x$, can be extended[2] in the $(2 \mid 2)$-graded field setup to the parity-preserving Hamiltonian operator [85],

$$\hat{P}_2 = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} & -u_2 & u_1 & 2u_0\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{0;x} \\ -u_2 & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + u_{12} & -2u_0\frac{\mathrm{d}}{\mathrm{d}x} - u_{0;x} & 3u_1\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{1;x} \\ u_1 & 2u_0\frac{\mathrm{d}}{\mathrm{d}x} + u_{0;x} & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + u_{12} & 3u_2\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{2;x} \\ 2u_0\frac{\mathrm{d}}{\mathrm{d}x} & -3u_1\frac{\mathrm{d}}{\mathrm{d}x} - u_{1;x} & -3u_2\frac{\mathrm{d}}{\mathrm{d}x} - u_{2;x} & \underline{\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + 4u_{12}\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{12;x}} \end{pmatrix}. \tag{3.3}$$

Here the fields u_0 and u_{12} are bosonic, u_1 and u_2 are fermionic together with their derivatives w.r.t. x. Likewise, the components $\psi_0 \simeq \delta\mathcal{H}/\delta u_0$ and $\psi_{12} \simeq \delta\mathcal{H}/\delta u_{12}$ of the columns $\vec{\psi} = {}^t\big(\psi_0, \psi_1, \psi_2, \psi_{12}\big)$ are even-graded and ψ_1, ψ_2 are odd-graded. The operator (3.3) is unique in the class of Hamiltonian total differential operators that merge to scalar N=2 super-operators which are local in $\mathcal{D}_i$ and whose coefficients depend on the super-field $\boldsymbol{u}$ and its super-derivatives, see (3.7) below. Operator (3.3) determines the N=2 classical super-conformal algebra [22]. Conversely, the Poisson bracket given by (3.3) reduces to the second Poisson bracket for (1.1), whenever one sets equal to zero the fields u_0, u_1, and u_2 both in the coefficients of (3.3) and in all Hamiltonians; the operator $\hat{A}_2^{\mathrm{KdV}}$ is underlined in (3.3).

[2]Likewise, we extend Gardner's deformation (2.1) of (1.1) to the deformation of two-component bosonic limit (3.10) for (1.2) with a=4. Hence we reproduce the conservation laws for (3.10) and, again, extend them to the bosonic super-Hamiltonians of full system (1.2).

By construction, Mathieu's extensions of Korteweg–de Vries equation (1.1) are determined by operator (3.3) and the bosonic Hamiltonian functional,

$$\mathcal{H}^{(2)} = \int \Big[u_0 u_{0;xx} \underline{- u_{12}^2} + u_1 u_{1;x} + u_2 u_{2;x} + a \cdot \big(u_0^2 u_{12} - 2 u_0 u_1 u_2 \big) \Big] \, \mathrm{d}x, \tag{3.4}$$

which incorporates $H^{(2)}_{\mathrm{KdV}}$ as the underlined term; similar to (3.7), Hamiltonian (3.4) will be realized by (3.6) as the bosonic N=2 super-Hamiltonian. Now we have that

$$u_{i;t} = \big(\hat{P}_2\big)_{ij} \big(\delta \mathcal{H}^{(2)} / \delta u_j\big), \qquad i, j \in \{0, 1, 2, 12\}.$$

This yields the system

$$\begin{aligned}
u_{0;t} &= -u_{0;xxx} + \big(a u_0^3 - (a+2) u_0 u_{12} + (a-1) u_1 u_2 \big)_x, && \text{(3.5a)} \\
u_{1;t} &= -u_{1;xxx} + \big(\quad (a+2) u_0 u_{2;x} + (a-1) u_{0;x} u_2 - 3 u_1 u_{12} + 3 a u_0^2 u_1 \big)_x, && \text{(3.5b)} \\
u_{2;t} &= -u_{2;xxx} + \big(-(a+2) u_0 u_{1;x} - (a-1) u_{0;x} u_1 - 3 u_2 u_{12} + 3 a u_0^2 u_2 \big)_x, && \text{(3.5c)} \\
\underline{u_{12;t}} &= \underline{-u_{12;xxx} - 6 u_{12} u_{12;x}} + 3 a u_{0;x} u_{0;xx} + (a+2) u_0 u_{0;xxx} && \\
&\qquad + 3 u_1 u_{1;xx} + 3 u_2 u_{2;xx} + 3 a \big(u_0^2 u_{12} - 2 u_0 u_1 u_2 \big)_x. && \text{(3.5d)}
\end{aligned}$$

Obviously, it retracts to (1.1), which we underline in (3.5), under the reduction $u_0 = 0$, $u_1 = u_2 = 0$.

At all $a \in \mathbb{R}$, Hamiltonian (3.4) equals

$$\boldsymbol{\mathcal{H}}^{(2)} = \int \big(\boldsymbol{u} \mathcal{D}_1 \mathcal{D}_2 (\boldsymbol{u}) + \tfrac{a}{3} \boldsymbol{u}^3 \big) \mathrm{d}\boldsymbol{\theta} \mathrm{d}x, \qquad \text{where } \mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta_1 \mathrm{d}\theta_2. \tag{3.6}$$

Likewise, the structure (3.3), which is independent of a, produces the N=2 super-operator

$$\hat{\boldsymbol{P}}_2 = \mathcal{D}_1 \mathcal{D}_2 \tfrac{\mathrm{d}}{\mathrm{d}x} + 2 \boldsymbol{u} \tfrac{\mathrm{d}}{\mathrm{d}x} - \mathcal{D}_1(\boldsymbol{u}) \mathcal{D}_1 - \mathcal{D}_2(\boldsymbol{u}) \mathcal{D}_2 + 2 \boldsymbol{u}_x. \tag{3.7}$$

Thus we recover Mathieu's super-equations (1.2) [84], which are Hamiltonian with respect to (3.7) and functional (3.6): $\boldsymbol{u}_t = \hat{\boldsymbol{P}}_2 \big(\frac{\delta}{\delta \boldsymbol{u}} (\boldsymbol{\mathcal{H}}_2) \big)$. In component notation, super-equations (1.2) are (3.5).

The assumption that, for a given a, super-system (1.2) admits infinitely many integrals of motion yields the triplet $a \in \{-2, 1, 4\}$, see [84]. The same values of a are exhibited by the Painlevé analysis for N=2 super-equations (1.2), see [17].

The three systems (1.2) have the common second Poisson structure, which is given by (3.7), but the three 'junior' first Hamiltonian operators $\hat{\boldsymbol{P}}_1$ for them do not coincide [85, 84, 58]. Moreover, system (1.2) with a=4 is radically different from the other two, both from the Hamiltonian and Lax viewpoints.

Proposition 3. The N=2 supersymmetric hierarchy of Mathieu's a=4 Korteweg–de Vries equation is bi-Hamiltonian with respect to local super-operator (3.7) and the junior Hamiltonian operator[3] $\hat{\boldsymbol{P}}_1^{a=4} = \mathrm{d}/\mathrm{d}x$, which is obtained from $\hat{\boldsymbol{P}}_2^{a=4}$ by the shift $\boldsymbol{u} \mapsto \boldsymbol{u} + \boldsymbol{\lambda}$ of the super-field $\boldsymbol{u}$, see [29, 115]:

$$\hat{\boldsymbol{P}}_1^{a=4} = \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\Big|_{\boldsymbol{\lambda}=0} \hat{\boldsymbol{P}}_2^{a=4}\Big|_{\boldsymbol{u}+\boldsymbol{\lambda}}.$$

The two operators are Poisson compatible and generate the tower of *nonlocal* higher structures $\hat{\boldsymbol{P}}_{k+2} = \big(\hat{\boldsymbol{P}}_2 \circ \hat{\boldsymbol{P}}_1^{-1}\big)^k \circ \hat{\boldsymbol{P}}_2$, $k \geq 1$, for the N=2, a=4–SKdV hierarchy, see [45, 76]. Although $\hat{\boldsymbol{P}}_3$ is nonlocal (c.f. [108]), its bosonic limits under (1.3) yield the *local* third Hamiltonian structure $\hat{A}_2$ for the Kaup–Boussinesq equation, which determines the evolution along the second time $t_2 \equiv \xi$ in the bosonic limit of the N=2, a=4–SKdV hierarchy (see Proposition 1 on p. 12).

Remark 2. The Kaup–Boussinesq system [18, 54] arising here is equivalent to the Kaup–Broer system (the difference amounts to notation). A bi-Hamiltonian N=2 super-extension of the latter is known from [83]. A tri-Hamiltonian two-fermion N=1 super-extension of the Kaup-Broer system was constructed in [19] such that in the bosonic limit the three known Hamiltonian structures for the initial system are recovered. At the same time, a boson-fermion N=1 super-extension of the Kaup–Broer equation with two local and the nonlocal third Hamiltonian structures was derived in [108]; seemingly, the latter equaled the composition $\hat{\boldsymbol{P}}_2 \circ \hat{\boldsymbol{P}}_1^{-1} \circ \hat{\boldsymbol{P}}_2$, but it remained to prove that the suggested nonlocal super-operator is skew-adjoint, that the bracket induced on the space of bosonic super-Hamiltonians does satisfy the Jacobi identity, and that the hierarchy flows produced by the nonlocal operator remain local.

There is a deep reason for the geometry of the a=4–SKdV to be exceptionally rich. All the three integrable N=2 supersymmetric KdV equations (1.2) admit the Lax representations $L_{t_3} = [A^{(3)}, L]$, see [13, 85, 99, 110]. For a=4, the four roots of the Lax operator $L_{a=4} = -(\mathcal{D}_1\mathcal{D}_2 + \boldsymbol{u})^2$, which are $\mathcal{L}_{1,\pm} = \pm\mathrm{i}(\mathcal{D}_1\mathcal{D}_2 + \boldsymbol{u})$, $\mathrm{i}^2 = -1$, and the super-pseudodifferential operators $\mathcal{L}_{2,\pm} = \pm\frac{\mathrm{d}}{\mathrm{d}x} + \sum_{i>0}(\cdots)\cdot\big(\frac{\mathrm{d}}{\mathrm{d}x}\big)^{-i}$, generate the odd-index flows of the SKdV hierarchy via $L_{t_{2k+1}} = [(\mathcal{L}_2^{2k+1})_{\geq 0}, L]$. In particular, we have $A_{a=4}^{(3)} = \big(L^{3/2}\big)_{\geq 0}$ mod $(\mathcal{D}_1\mathcal{D}_2 + \boldsymbol{u})^3$. However, the *entire* a=4 hierarchy is reproduced in the Lax form via $(\mathcal{L}_1^k\mathcal{L}_2)_{t_\ell} = \big[\big(\mathcal{L}_1^\ell\mathcal{L}_2\big)_{\geq 0}, \mathcal{L}_1^k\mathcal{L}_2\big]$ for all $k \in \mathbb{N}$, c.f. [81]. Hence the super-residues[4] of the operators $\mathcal{L}_1^k\mathcal{L}_2$ are conserved.

Consequently, unlike the other two, super-equation (1.2) with a=4 admits twice as many constants of motion as there are for the super-equations with a=−2 or a=1. For

[3] The nonzero entries of the (4×4)-matrix representation $\hat{P}_1$ for the Hamiltonian super-operator $\hat{\boldsymbol{P}}_1^{a=4}$ are $\big(\hat{P}_1\big)_{0,12} = \big(\hat{P}_1\big)_{2,1} = \big(\hat{P}_1\big)_{12,0} = -\big(\hat{P}_1\big)_{1,2} = \mathrm{d}/\mathrm{d}x$.

[4] We recall that the N=2 super-residue $\operatorname{Sres} M$ of a super-pseudodifferential operator M is the coefficient of $\mathcal{D}_1\mathcal{D}_2 \circ \big(\frac{\mathrm{d}}{\mathrm{d}x}\big)^{-1}$ in M.

convenience, let us recall that super-equations (1.2) are homogeneous with respect to the weights $|\mathrm{d}/\mathrm{d}x| \equiv 1$, $|\boldsymbol{u}| = 1$, $|\mathrm{d}/\mathrm{d}t| = 3$. Hence we conclude that, for each nonnegative integer k, there appears the nontrivial conserved density $\operatorname{Sres} \mathcal{L}_1^k \mathcal{L}_2$, see above, of weight $k+1$. The even weights also enter the play. Consequently, there are twice as many commuting super-flows assigned to the twice as many Hamiltonians.

Example 5. The additional super-Hamiltonian $\boldsymbol{\mathcal{H}}^{(1)} = \frac{1}{2}\int \boldsymbol{u}^2 \mathrm{d}\boldsymbol{\theta}\mathrm{d}x$ for (1.2) with a=4, and the second structure (3.7), — or, equivalently, the first operator $\hat{\boldsymbol{P}}_1 = \mathrm{d}/\mathrm{d}x$ and the Hamiltonian $\boldsymbol{\mathcal{H}}^{(2)}$, or $\hat{\boldsymbol{P}}_3$ and $\boldsymbol{\mathcal{H}}^{(0)} = \int \boldsymbol{u}\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}x$, see above, — generate the N=2 supersymmetric equation

$$\boldsymbol{u}_\xi = \mathcal{D}_1\mathcal{D}_2\boldsymbol{u}_x + 4\boldsymbol{u}\boldsymbol{u}_x = \hat{\boldsymbol{P}}_3\left(\frac{\delta}{\delta \boldsymbol{u}}(\boldsymbol{\mathcal{H}}^{(0)})\right) = \hat{\boldsymbol{P}}_2\left(\frac{\delta}{\delta \boldsymbol{u}}(\boldsymbol{\mathcal{H}}^{(1)})\right) = \hat{\boldsymbol{P}}_1\left(\frac{\delta}{\delta \boldsymbol{u}}(\boldsymbol{\mathcal{H}}^{(2)})\right), \quad \xi \equiv t_2. \tag{3.8}$$

Super-equation (3.8) was referred to as the N=2 'Burgers' equation in [60, 72] due to the recovery of $\boldsymbol{u}_\xi = \boldsymbol{u}_{xx} + 4\boldsymbol{u}\boldsymbol{u}_x$ on the diagonal $\theta_1 = \theta_2$.

In components, the N=2 super-equation (3.8) reads

$$\begin{aligned} u_{0;\xi} &= \bigl(-u_{12} + 2u_0^2\bigr)_x, & u_{1;\xi} &= \bigl(u_{2;x} + 4u_0u_1\bigr)_x, \\ u_{2;\xi} &= \bigl(-u_{1,x} + 4u_0u_2\bigr)_x, & u_{12;\xi} &= \bigl(u_{0;xx} + 4u_0u_{12} - 4u_1u_2\bigr)_x. \end{aligned}$$

Clearly, it admits reduction (1.3); moreover, Kaup–Boussinesq system (1.4) (see [18, 54] or [52, 82, 105] and references therein) is the only possible limit for (3.8),

$$u_{0;\xi} = \bigl(-u_{12} + 2u_0^2\bigr)_x, \qquad u_{12;\xi} = \bigl(u_{0;xx} + 4u_0u_{12}\bigr)_x. \tag{1.4}$$

System (1.4) is equivalent to the Kaup–Broer equation via an invertible substitution. In these terms, super-equation (3.8) is a super-extension of the Kaup–Boussinesq system [19, 83, 108]. In their turn, the first three Poisson structures for (1.2) with a=4 are reduced under (1.3) to the respective *local* structures for (1.4), see Proposition 1 on p. 12.

Our interest in the recursive production of the integrals of motion for (1.2) grew after the discovery, see [60], of new n-soliton solutions,

$$\boldsymbol{u} = \mathsf{A}(a)\cdot\mathcal{D}_1\mathcal{D}_2\log\Bigl(1 + \sum_{i=1}^{n}\alpha_i\exp\bigl(k_ix - k_i^3\cdot t \pm \mathrm{i}\,k_i\cdot\theta_1\theta_2\bigr)\Bigr), \qquad \mathsf{A}(a) = \begin{cases}1, & a=1,\\ \frac{1}{2}, & a=4,\end{cases} \tag{3.9}$$

for super-equations (1.2) with a=1 or a=4 (but not a=-2 or any other $a \in \mathbb{R}\setminus\{1,4\}$). In formula (3.9), the wave numbers $k_i \in \mathbb{R}$ are arbitrary, and the phases α_i can be rescaled to $+1$ for non-singular n-soliton solutions by appropriate shifts of n higher times in the SKdV hierarchy. A spontaneous decay of fast solitons and their transition into the virtual states, on the emerging background of previously invisible, slow solitons, look paradoxical for such KdV-type systems (a=1 or a=4), since they possess an infinity of integrals of motion.

New solutions (3.9) of (1.2) with a=1 or a=4 are subject to condition (1.3) and therefore they satisfy the bosonic limits of these N=2 super-systems. In the same way, bosonic limit (1.4) of (3.8) admits multi-soliton solutions in Hirota's form (3.9), now with the exponents $\eta_i = k_i x \pm \mathrm{i}k_i^2\xi \pm \mathrm{i}k_i\theta_1\theta_2$, see [60]. This makes the role of such two-component bosonic reductions particularly important. We recall that reduction (1.3) of (1.2) with a=1 yields the Kersten–Krasil'shchik equation, see [57] or [60] and references therein. In Chapter 2 we considered the bosonic limit of the N=2, a=4 SKdV equation,

$$u_{0;t} = -u_{0;xxx} + 12u_0^2 u_{0;x} - 6\big(u_0 u_{12}\big)_x, \tag{3.10a}$$

$$u_{12;t} = -u_{12;xxx} - 6u_{12}u_{12;x} + 12u_{0;x}u_{0;xx} + 6u_0 u_{0;xxx} + 12\big(u_0^2 u_{12}\big)_x, \tag{3.10b}$$

which succeeds Kaup–Boussinesq equation (1.4) in its tri-Hamiltonian hierarchy. We shall construct a new Gardner deformation for it (c.f. [52]) in section 2.3.

In general, system (3.5) with a=4 admits three one-component reductions (except $u_0 \not\equiv 0$) and three two-component reductions, which are indicated by the edges that connect the remaining components in the diagram

$$\begin{array}{ccccc} & & u_0 & & \\ & & \| & & \\ u_1 & = & u_{12} & = & u_2. \end{array}$$

System (3.5) with a=4 has no three-component reductions obtained by setting to zero only one of the four fields in (3.2). We conclude this entire Chapter 3 by presenting a Gardner deformation for the two-component boson-fermion reduction $u_0 \equiv 0$, $u_2 \equiv 0$ of the N=2, a=4–SKdV system, see (3.14) on p. 39.

3.4 Deformation problem for N=2, a=4–SKdV equation

In this section we formulate the two-step algorithm for a recursive production of the bosonic super-Hamiltonians $\boldsymbol{\mathcal{H}}^{(k)}[\boldsymbol{u}]$ for the N=2 supersymmetric a=4–SKdV hierarchy. Essentially, we convert the geometric problem to an explicit computational procedure. Our scheme can be applied to other KdV-type super-systems (in particular, to (1.2) with a= − 2 or a=1).

We split the Gardner deformation problem for the N=2 supersymmetric hierarchy of (1.2) with a=4 in two main and several auxiliary steps.

First, we note that Miura's contraction $\mathfrak{m}_\varepsilon\colon \mathcal{E}(\varepsilon) \to \mathcal{E}$, which encodes the recurrence relation between the conserved densities, is common for all equations of the hierarchy. Indeed, the densities (and hence any differential relations between them) are shared by all the equations. Therefore, we pass to the deformation problem for the N=2 super-Burgers equation (3.8). This makes the first simplification of the Gardner deformation problem for the N=2, a=4 super-KdV hierarchy.

Second, let $\boldsymbol{h}^{(k)}$ be an N=2 super-conserved density for an evolutionary super-equation $\mathcal{E}$, meaning that its velocity w.r.t. a time τ, $\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{h}^{(k)} = \mathcal{D}_1(\dots) + \mathcal{D}_2(\dots)$, is a total divergence on $\mathcal{E}$. By definition of $\mathcal{D}_i$, see (1.2), the $\theta_1\theta_2$-component $h_{12}^{(k)}$ of such $\boldsymbol{h}^{(k)} = h_0^{(k)} + \theta_1 \cdot h_1^{(k)} + \theta_2 \cdot h_2^{(k)} + \theta_1\theta_2 \cdot h_{12}^{(k)}$ is conserved in the classical sense, $\frac{\mathrm{d}}{\mathrm{d}\tau}h_{12}^{(k)} = \frac{\mathrm{d}}{\mathrm{d}x}(\dots)$ on $\mathcal{E}$. Let us consider the correlation between the conservation laws for the full N=2 super-system $\mathcal{E}$ and for its reductions that are obtained by setting certain component(s) of $\boldsymbol{u}$ to zero. In what follows, we study bosonic reduction (1.3). Other reductions of super-equation (1.2) are discussed in section 3.5, see (3.13) on p. 38.

We suppose that the bosonic limit $\lim_B \mathcal{E}$ of the super-equation $\mathcal{E}$ exists, which is the case for (1.2) and (3.8). By the above, each conserved super-density $\boldsymbol{h}^{(k)}[\boldsymbol{u}]$ determines the conserved density $h_{12}^{(k)}[u_0, u_{12}]$, which may become trivial. As in [13], we assume that the super-system $\mathcal{E}$ does not admit any conserved super-densities that vanish under reduction (1.3). Then, for such $h_{12}^{(k)}$ that originates from $\boldsymbol{h}^{(k)}$ by construction, the equivalence class $\{\boldsymbol{h}^{(k)} \mod \operatorname{im}\mathcal{D}_i\}$ is uniquely determined by

$$\int h_{12}^{(k)}[u_0, u_{12}]\,\mathrm{d}x = \int \boldsymbol{h}^{(k)}[\boldsymbol{u}]\big|_{u_1=u_2=0}\mathrm{d}\boldsymbol{\theta}\mathrm{d}x, \qquad \text{here } N{=}2 \text{ and } \mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Berezin's definition of a super-integration, $\int \mathrm{d}\theta_i = 0$ and $\int \theta_i\,\mathrm{d}\theta_i = 1$, implies that the problem of recursive generation of the N=2 super-Hamiltonians $\boldsymbol{\mathcal{H}}^{(k)} = \int \boldsymbol{h}^{(k)}\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}x$ for the SKdV hierarchy amounts to the generation of the equivalence classes $\int h_{12}^{(k)}\,\mathrm{d}x$ for the respective $\theta_1\theta_2$-component. We conclude that a solution of Gardner's deformation problem for supersymmetric system (3.8) may not be subject to the supersymmetry invariance. This is a key point to reasonings.

We stress that the equivalence class of such functions $h_{12}^{(k)}[u_0, u_{12}]$ that originate from $\boldsymbol{\mathcal{H}}^{(k)}$ by (1.3) is, generally, much more narrow than the equivalence class $\{h_{12}^{(k)} \mod \operatorname{im}\mathrm{d}/\mathrm{d}x\}$ of all conserved densities for the bosonic limit $\lim_B \mathcal{E}$. Obviously, there are differential functions of the form $\frac{\mathrm{d}}{\mathrm{d}x}\big(f[u_0, u_{12}]\big)$ that can not be obtained[5] as the $\theta_1\theta_2$-component of any $\big[\mathcal{D}_1(\cdot) + \mathcal{D}_2(\cdot)\big]\big|_{u_1=u_2=0}$, which is trivial in the super-sense. Therefore, let $h_{12}^{(k)}$ be *any* recursively given sequence of integrals of motion for $\lim_B \mathcal{E}$ (*e.g.*, suppose that they are the densities of the Hamiltonians $\mathcal{H}^{(k)}$ for the hierarchy of $\lim_B \mathcal{E}$), and let it be known that each $\mathcal{H}^{(k)} = \int h_{12}^{(k)}\,\mathrm{d}x$ does correspond to the super-analogue $\boldsymbol{\mathcal{H}}^{(k)} = \int \boldsymbol{h}^{(k)}\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}x$. Then the reconstruction of $\boldsymbol{h}^{(k)}$ requires an intermediate step, which is the elimination of excessive, homologically trivial terms under $\mathrm{d}/\mathrm{d}x$ that preclude a given $h_{12}^{(k)}$ to be extended to the full super-density in terms of the N=2 super-field $\boldsymbol{u}$. This is illustrated in section 3.5.

[5] Under the assumption of weight homogeneity, the freedom in the choice of such $f[u_0, u_{12}]$ is decreased, but the gap still remains.

Thirdly, the gap between the two types of equivalence for the integrals of motion manifests the distinction between the deformations $\big(\lim_B \mathcal{E}\big)(\varepsilon)$ of bosonic limits and, on the other hand, the bosonic limits $\lim_B \mathcal{E}(\varepsilon)$ of N=2 super-deformations. The two operations, Gardner's extension of $\mathcal{E}$ to $\mathcal{E}(\varepsilon)$ and taking the bosonic limit $\lim_B \mathcal{F}$ of an equation $\mathcal{F}$, are not permutable. The resulting systems can be different. Namely, according to the classical scheme ([102], [59]), *each* equation in the evolutionary system $\big(\lim_B \mathcal{E}\big)(\varepsilon)$ represents a conserved current, whence each Taylor coefficient of the respective field is conserved, see Example 1. At the same time, for $\lim_B \mathcal{E}(\varepsilon)$, the conservation is required only for the field $\tilde{u}_{12}(\varepsilon)$, which is the $\theta_1\theta_2$-component of the extended super-field $\tilde{\boldsymbol{u}}(\varepsilon)$. Other equations in $\lim_B \mathcal{E}(\varepsilon)$ can have any form.[6]

In this notation, we strengthen the problem of recursive generation of the super-Hamiltonians for the N=2 super-equation (3.8). Namely, in section 2.3 we constructed true Gardner's deformations for its two-component bosonic limit (1.4). The solution to the Gardner deformation problem generates the recurrence relation between the nontrivial conserved densities $h_{12}^{(k)}$ which, in the meantime, depend on u_0 and u_{12}. By correlating them with the $\theta_1\theta_2$-components of the super-densities $\boldsymbol{h}^{(k)}$ that depend on $\boldsymbol{u}$, we derive the Hamiltonians $\boldsymbol{\mathcal{H}}^{(k)}$, $k \geq 0$, for the N=2 supersymmetric a=4–KdV hierarchy, see section 3.5.

3.5 Super-Hamiltonians for N=2, a=4–SKdV hierarchy

In this section we assign the bosonic super-Hamiltonians $\boldsymbol{\mathcal{H}}^{(k)} = \int \boldsymbol{h}^{(k)}[\boldsymbol{u}]\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}x$ of (1.2) with a=4 to the Hamiltonians $H^{(k)} = \int h_{12}^{(k)}[u_0, u_{12}]\,\mathrm{d}x$ of its bosonic limit (3.10). Also, we establish the no-go result on the super-field, N=2 supersymmetry invariant deformations of a=4–SKdV that would retract to (2.1) under the respective reduction in super-field (3.2). At the same time, we initiate the study of Gardner's deformations for reductions of (3.5) other than (1.3), and here we find the deformations of two-component fermion-boson limit in it. However, we observe that the new solutions can not be merged with the deformation (2.5) for the bosonic limit of (3.5).

From Section 2.3 we know the procedure for recursive production of the Hamiltonians $H^{(k)} = \int h^{(k)}\,\mathrm{d}x$ for bosonic limit (3.10) of the N=2, a=4–SKdV equation, here $h^{(2k)} = \tilde{u}_0^{(2k)}$ and $h^{(2k+1)} = \tilde{u}_{12}^{(2k)}$. In section 3.4, we explained why the reconstruction of the densities $\boldsymbol{h}^{(k)}$ for the bosonic super-Hamiltonians $\boldsymbol{\mathcal{H}}^{(k)}$ from $h^{(k)}\big[u_0, u_{12}\big]$ requires an intermediate step. Namely, it amounts to the proper choice of the representatives $h_{12}^{(k)}$ within the equivalence

[6] Still, the four components of the original N=2 supersymmetric equations within the hierarchy of (1.2) *are* written in the form of conserved currents. A helpful counter-example, Gardner's extension of the $N=1$ super-KdV equation, is discussed in [84, 98].

class $\{h^{(k)} \mod \operatorname{im} \frac{\mathrm{d}}{\mathrm{d}x}\}$ such that $h_{12}^{(k)}$ can be realized under (1.3) as the $\theta_1\theta_2$-component of the super-density $\boldsymbol{h}^{(k)}$. This allows us to restore the dependence on the components u_1 and u_2 of (3.2) and to recover the supersymmetry invariance. The former means that each $\boldsymbol{h}^{(k)}$ is conserved on (3.5) and the latter implies that $\boldsymbol{h}^{(k)}$ becomes a differential function in $\boldsymbol{u}$.

The correlation between *unknown* bosonic super-differential polynomials $\boldsymbol{h}^{(k)}[\boldsymbol{u}]$ and the densities $h^{(k)}[u_0, u_{12}]$, which are produced by the recurrence relation, is established as follows. First, we generate the homogeneous super-differential polynomial ansatz for the bosonic $\boldsymbol{h}^{(k)}$ using `GenSSPoly`. Second, we split the super-field $\boldsymbol{u}$ using the right-hand side of (3.2) and obtain the $\theta_1\theta_2$-component $h_{12}^{(k)}[u_0, u_1, u_2, u_{12}]$ of the differential function $\boldsymbol{h}^{(k)}[\boldsymbol{u}]$. This is done by the procedure `ToCoo`. which is also available in SsTools [73, 126]. Thirdly, we set to zero the components u_1 and u_2 of the super-field $\boldsymbol{u}$. This gives the ansatz $h_{12}^{(k)}[u_0, u_{12}]$ for the representative of the conserved density in the vast equivalence class. By the above, the gap between $h_{12}^{(k)}$ and the known $h^{(k)}$ amounts to $\frac{\mathrm{d}}{\mathrm{d}x}(f^{(k)})$, where $f^{(k)}[u_0, u_{12}]$ is a homogeneous differential polynomial. We remark that the choice of f is not unique due to the freedom in the choice of $\boldsymbol{h}^{(k)} \mod \mathcal{D}_1(\dots)+\mathcal{D}_2(\dots)$. We thus arrive at the linear algebraic equation

$$h_{12}^{(k)} - \tfrac{\mathrm{d}}{\mathrm{d}x} f^{(k)} = h^{(k)}, \tag{3.11}$$

which implies the equality of the respective coefficients in the polynomials. The homogeneous polynomial ansatz for $f^{(k)}$ is again generated by `GenSSPoly`. Then equation (3.11) is split to the algebraic system by SsTools and solved by Crack [125]. Hence we obtain the coefficients in $h_{12}^{(k)}$ and $f^{(k)}$. *A posteriori*, the freedom in the choice of $f^{(k)}$ is redundant, and it is convenient to set the surviving *unassigned* coefficients to zero. Indeed, they originate from the choice of a representative from the equivalence class for the super-density $\boldsymbol{h}^{(k)}[\boldsymbol{u}]$. This concludes the algorithm for the recursive production of homogeneous bosonic $N{=}2$ supersymmetry-invariant super-Hamiltonians $\boldsymbol{\mathcal{H}}^{(k)}$ for the $N{=}2$, $a{=}4$–SKdV hierarchy.

Example 6. Let us reproduce the first seven super-Hamiltonians for (1.2), which were found in [84]. In contrast with Example 2, we now list the *properly chosen* representatives $h_{12}^{(k)}[u_0, u_{12}]$ for the equivalence classes of conserved densities $\tilde{u}_0^{(2k)}$ and $\tilde{u}_{12}^{(2k)}$, here $k \leq 3$. Then we expose the conserved super-densities $\boldsymbol{h}^{(k)}$ such that the respective expressions $h_{12}^{(k)}$ are obtained from the $\theta_1\theta_2$-components $\int \boldsymbol{h}^{(k)}\,\mathrm{d}\boldsymbol{\theta}$ by reduction (1.3).

$$h_{12}^{(0)} = u_0 \sim \tilde{u}_0^{(0)}, \qquad \boldsymbol{h}^{(0)} = -\mathcal{D}_1\mathcal{D}_2(\boldsymbol{u}) \sim 0, \tag{3.12a}$$

$$h_{12}^{(1)} = u_{12} \sim \tilde{u}_{12}^{(0)}, \qquad \boldsymbol{h}^{(1)} = \boldsymbol{u}, \tag{3.12b}$$

$$h_{12}^{(2)} = -2u_{12}u_0 \sim \tilde{u}_0^{(2)}, \qquad \boldsymbol{h}^{(2)} = \boldsymbol{u}^2, \tag{3.12c}$$

$$h_{12}^{(3)} = \tfrac{3}{4}u_{12}^2 - 3u_{12}u_0^2 + \tfrac{3}{4}u_{0;x}^2 \sim \tilde{u}_{12}^{(2)}, \qquad \boldsymbol{h}^{(3)} = \boldsymbol{u}^3 - \tfrac{3}{4}\boldsymbol{u}\mathcal{D}_1\mathcal{D}_2(\boldsymbol{u}), \tag{3.12d}$$

$$
\begin{aligned}
h_{12}^{(4)} &= 3u_{12}^2u_0 - 4u_{12}u_0^3 - \tfrac{3}{2}u_0^2u_{0;xx} - u_{12;x}u_{0;x} \sim \tilde{u}_0^{(4)},\\
\boldsymbol{h}^{(4)} &= \boldsymbol{u}^4 - \tfrac{1}{2}\boldsymbol{u}\boldsymbol{u}_{xx} - \tfrac{3}{2}\boldsymbol{u}^2\mathcal{D}_1\mathcal{D}_2(\boldsymbol{u}), && (3.12e)\\
h_{12}^{(5)} &= -\tfrac{5}{4}u_{12}^3 + \tfrac{15}{2}u_{12}^2u_0^2 - 5u_{12}u_0^4 + 5u_{12}u_0u_{0;xx} + \tfrac{15}{8}u_{12}u_{0;x}^2 + \tfrac{15}{2}u_0^2u_{0;x}^2 + \tfrac{5}{16}u_{12;x}^2 +\\
&\quad + \tfrac{5}{16}u_{0;xx}^2 \sim \tilde{u}_{12}^{(4)}, \qquad \boldsymbol{h}^{(5)} = \boldsymbol{u}^5 - \tfrac{15}{16}\boldsymbol{u}^2\boldsymbol{u}_{xx} + \tfrac{5}{8}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^2\boldsymbol{u} - \tfrac{5}{2}\boldsymbol{u}^3\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}, && (3.12f)\\
h_{12}^{(6)} &= -\tfrac{15}{4}u_{12}^3u_0 + 15u_{12}^2u_0^3 - \tfrac{15}{8}u_{12}^2u_{0;xx} - 6u_{12}u_0^5 - \tfrac{75}{4}u_{12}u_0u_{0;x}^2 - \tfrac{3}{8}u_{12}u_{0;xxxx} +\\
&\quad + 5u_0^3u_{12;xx} + 15u_0^3u_{0;x}^2 + \tfrac{15}{8}u_0u_{12;x}^2 + \tfrac{15}{8}u_0u_{0;xx}^2 \sim \tilde{u}_0^{(6)},\\
\boldsymbol{h}^{(6)} &= \boldsymbol{u}^6 - \tfrac{15}{8}\boldsymbol{u}^3\boldsymbol{u}_{xx} + \tfrac{3}{16}\boldsymbol{u}\boldsymbol{u}_{4x} + \tfrac{15}{8}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^2 - \tfrac{15}{4}\boldsymbol{u}^4\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} + \tfrac{15}{8}\boldsymbol{u}_{xx}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} -\\
&\quad + \tfrac{5}{8}\mathcal{D}_1\mathcal{D}_2(\boldsymbol{u})\mathcal{D}_1(\boldsymbol{u})\mathcal{D}_1(\boldsymbol{u}_x), && (3.12g)\\
h_{12}^{(7)} &= -\tfrac{21}{8}u_{0;4x}u_0u_{12} + \tfrac{7}{64}u_{0;xxx}^2 + \tfrac{105}{16}u_{0;xx}^2u_0^2 + \tfrac{35}{32}u_{0;xx}^2u_{12} - \tfrac{105}{8}u_{0;xx}u_0u_{12}^2 - \tfrac{105}{64}u_{0;4x}^4 -\\
&\quad - \tfrac{35}{16}u_{0;x}^2u_{12;xx} + \tfrac{105}{4}u_{0;x}^2u_0^4 - \tfrac{525}{8}u_{0;x}^2u_0^2u_{12} - \tfrac{175}{32}u_{0;x}^2u_{12}^2 + \tfrac{7}{64}u_{12;xx}^2 + \tfrac{35}{4}u_{12;xx}u_0^4 +\\
&\quad + \tfrac{105}{16}u_{12;x}^2u_0^2 - \tfrac{35}{32}u_{12;x}^2u_{12} - 7u_0^6u_{12} + \tfrac{105}{4}u_0^4u_{12}^2 - \tfrac{105}{8}u_0^2u_{12}^3 + \tfrac{35}{64}u_{12}^4 \sim \tilde{u}_{12}^{(6)},\\
\boldsymbol{h}^{(7)} &= \boldsymbol{u}^7 - \tfrac{105}{32}\boldsymbol{u}^3\boldsymbol{u}_{xx} + \tfrac{7}{32}\boldsymbol{u}^2\boldsymbol{u}_{4x} - \tfrac{35}{64}\boldsymbol{u}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^3 + \tfrac{35}{8}\boldsymbol{u}^3(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^2 - \tfrac{35}{64}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})^2\boldsymbol{u}_{xx} -\\
&\quad - \tfrac{21}{4}\boldsymbol{u}^5\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} + \tfrac{105}{16}\boldsymbol{u}^2\boldsymbol{u}_{xx}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} + \tfrac{315}{64}\boldsymbol{u}\boldsymbol{u}_x^2\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} + \tfrac{35}{16}\boldsymbol{u}(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u})(\mathcal{D}_1\boldsymbol{u})(\mathcal{D}_1\boldsymbol{u}_x) -\\
&\quad - \tfrac{7}{64}\boldsymbol{u}_{4x}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u} - \tfrac{7}{8}\boldsymbol{u}(\mathcal{D}_1\boldsymbol{u}_{xx})(\mathcal{D}_1\boldsymbol{u}_x). && (3.12h)
\end{aligned}
$$

Of course, our super-densities $\boldsymbol{h}^{(k)}$ are equivalent to those in [84] up to adding trivial terms $\mathcal{D}_1(\dots) + \mathcal{D}_2(\dots)$.

Remark 3. Until now, we have not yet reported any attempt of construction of Gardner's *super-field* deformation for (1.2), which means that the ansatz for $\mathfrak{m}_\varepsilon$ and $\mathcal{E}(\varepsilon)$ is written in super-functions of $\boldsymbol{u}$ (c.f. [84]). This would yield the super-Hamiltonians $\boldsymbol{\mathcal{H}}^{(k)}$ at once, and the intermediate deformation (2.5) of a reduction (1.3) for (1.2) would not be necessary. At the same time, the knowledge of Gardner's deformations for the reductions allows to inherit a part of the coefficients in the super-field ansatz by fixing them in the component expansions (*e.g.*, see (2.1), (2.3), and (2.5)).

Unfortunately, this cut-through does not work for the N=2, a=4–SKdV equation.

Theorem 3 (N=2, a=4 'no go' [47]). Under the assumptions that N=2 supersymmetry-invariant Gardner's deformation $\mathfrak{m}_\varepsilon \colon \mathcal{E}(\varepsilon) \to \mathcal{E}$ of (1.2) with a=4 is regular at $\varepsilon = 0$, is scaling-homogeneous, and retracts to (2.1) under the reduction $u_0 = 0$, $u_1 = u_2 = 0$ in super-field (3.2), there is no such deformation.

This rigidity statement, although under a principally different set of initial hypotheses, is contained in [84]. In particular, there it was supposed that $\deg \mathfrak{m}_\varepsilon = \deg \mathcal{E}(\varepsilon) = 2$, which turns to be on the obstruction threshold, see below. We reveal the general nature of this 'no go' result.

Proof. Suppose there is the super-field Miura contraction $\mathfrak{m}_\varepsilon$,

$$\boldsymbol{u} = \tilde{\boldsymbol{u}} + \varepsilon\bigl(p_3\tilde{u}^2 - p_1\mathcal{D}_1\mathcal{D}_2\tilde{\boldsymbol{u}} + p_2\tilde{\boldsymbol{u}}_x\bigr) + \varepsilon^2\Bigl(p_{15}\tilde{\boldsymbol{u}}^3 + p_{13}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}_x + p_{10}\mathcal{D}_2(\tilde{\boldsymbol{u}})\mathcal{D}_1(\tilde{\boldsymbol{u}}) - p_{12}\mathcal{D}_1\mathcal{D}_2(\tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}} - p_{11}\mathcal{D}_1\mathcal{D}_2(\tilde{\boldsymbol{u}}_x) + p_{14}\tilde{\boldsymbol{u}}_{xx}\Bigr) + \cdots .$$

To recover deformation (2.1) upon u_{12} in $\boldsymbol{u}$, we split $\mathfrak{m}_\varepsilon$ in components and fix the coefficients of $\varepsilon\tilde{u}_{12;x}$ and $\varepsilon^2\tilde{u}_{12}^2$, see (2.1a). By this argument, the expansion of $\tilde{\boldsymbol{u}}_x$ yields $p_2 = 1$, while the equality $-p_{12}\mathcal{D}_1\mathcal{D}_2(\tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}} + p_{10}\mathcal{D}_2(\tilde{\boldsymbol{u}})\mathcal{D}_1(\tilde{\boldsymbol{u}}) = (p_{12} - p_{10})\theta_1\theta_2u_{12}^2 + \ldots$ implies that $p_{12} = p_{10} - 1$. Next, we generate the homogeneous ansatz for $\mathcal{E}(\varepsilon)$, which contains $\tilde{\boldsymbol{u}}_t = \cdots + \varepsilon^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\bigl(q_{17}(\mathcal{D}_2\boldsymbol{u})(\mathcal{D}_1\boldsymbol{u})\boldsymbol{u} + \ldots\bigr) + \ldots$ in the right-hand side (the coefficient q_{17} will appear in the obstruction). We stress that now both $\mathfrak{m}_\varepsilon$ and $\mathcal{E}(\varepsilon)$ can be formal power series in ε without any finite-degree polynomial truncation.

Now we split the determining equation $\mathfrak{m}_\varepsilon\colon \mathcal{E}(\varepsilon) \to \mathcal{E}$ to the sequence of super-differential polynomial equalities ordered by the powers of ε. By the regularity assumption, the coefficients of higher powers of ε never contribute to the equations that arise at its lower degrees. Consequently, every contradiction obtained at a finite order in the algebraic system is universal and precludes the existence of a solution. (Of course, we assume that the contradiction is not created artificially by an excessively low order polynomial truncation of the expansions in ε.)

This is the case for the N=2, a=4–SKdV. Using CRACK [125], we solve all but two algebraic equations in the quadratic approximation. The remaining system is

$$q_{17} = -p_{10}, \qquad p_{10} + q_{17} + 1 = 0.$$

This contradiction concludes the proof. □

Remark 4. In Theorem 3 for (1.2) with a=4, we state the non-existence of the Gardner deformation in a class of differential super-polynomials in $\boldsymbol{u}$, that is, of N=2 supersymmetry-invariant solutions that incorporate (2.1). Still, we do *not* claim the non-existence of local regular Gardner's deformations for the four-component system (3.5) in the class of differential functions of u_0, u_1, u_2, and u_{12}.

Consequently, it is worthy to deform the reductions of (3.5) other than (1.3). Clearly, if there is a deformation for the entire system, then such partial solutions contribute to it by fixing the parts of the coefficients.

Example 7. Let us consider the reduction $u_0 = 0$, $u_2 = 0$ in (3.5) with a=4. This is the two-component boson-fermion system

$$u_{1;t} = -u_{1;xxx} - 3\bigl(u_1u_{12}\bigr)_x, \qquad u_{12;t} = -u_{12;xxx} - 6u_{12}u_{12;x} + 3u_1u_{1;xx}. \tag{3.13}$$

Notice that system (3.13) is *quadratic*-nonlinear in both fields, hence the balance $\deg \mathfrak{m}_\varepsilon : \deg \mathcal{E}(\varepsilon)$ for its polynomial Gardner's deformations remains 1 : 1.

We found a unique Gardner's deformation of degree ≤ 4 for (3.13): the Miura contraction $\mathfrak{m}_\varepsilon$ is cubic in ε,

$$u_1 = \tilde{u}_1, \qquad u_{12} = \tilde{u}_{12} - \tfrac{1}{9}\varepsilon^3 \tilde{u}_1 \tilde{u}_{1;xx}, \tag{3.14a}$$

and the extension $\mathcal{E}(\varepsilon)$ is given by the formulas

$$\begin{aligned} \tilde{u}_{1;t} &= -\tilde{u}_{1;xxx} - 3\big(\tilde{u}_1 \tilde{u}_{12}\big)_x, \\ \tilde{u}_{12;t} &= -\tilde{u}_{12;xxx} - 6\tilde{u}_{12}\tilde{u}_{12;x} + 3\tilde{u}_1\tilde{u}_{1;xx} + \\ &\quad + \tfrac{1}{3}\varepsilon^3 \Big(u_1 u_{1;xx} u_{12} - 3u_1 u_{1;x} u_{12;x} + u_{1;x} u_{1;xxx}\Big)_x. \end{aligned} \tag{3.14b}$$

However, we observe, first, that contraction (2.1a) is not recovered[7] by (3.14a) under $u_1 \equiv 0$. Hence deformation (3.14) and its mirror copy under $u_1 \leftrightarrow -u_2$ can not be merged with (2.3) and (2.5) to become parts of the deformation for (3.5).

Second, we recall that the fields u_1 and u_2 are, seemingly, the only local fermionic conserved densities for (3.5) with a=4. Consequently, either the velocities $\tilde{u}_{1;t}$ and $\tilde{u}_{2;t}$ in Gardner's extensions $\mathcal{E}(\varepsilon)$ of (3.5) are not expressed in the form of conserved currents (although this is indeed so at $\varepsilon = 0$) or the components $u_i = u_i\big([\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}_{12}], \varepsilon\big)$ of the Miura contractions $\mathfrak{m}_\varepsilon$ are the identity mappings $u_i = \tilde{u}_i$, here $i = 1, 2$, whence either the Taylor coefficients $\tilde{u}_i^{(k)}$ of $\tilde{u}_i$ are not termwise conserved on (3.5) or there appear no recurrence relations at all.

We obtained the no-go statement for regular, scaling-homogeneous polynomial Gardner's deformations of the N=2, a=4–SKdV equation under the assumption that the solutions retract to original formulas (2.1) by Gardner [102].

We exposed the two-step procedure for recursive production of the bosonic super-Hamiltonians $\boldsymbol{\mathcal{H}}^{(k)}$. We formulated the entire algorithm in full detail such that, with elementary modifications, it is applicable to other supersymmetric KdV-type systems.

[7]Surprisingly, quadratic approximation (2.1a) in the deformation problem for (3.5) is very restrictive and leads to a unique solution (2.3)–(2.5) for (3.10). Relaxing this constraint and thus permitting the coefficient of $\varepsilon^2 \tilde{u}_{12}^2$ in $\mathfrak{m}_\varepsilon$ be arbitrary, we obtain two other real and two pairs of complex conjugate solutions for the deformations problem. They constitute the real and the complex orbit, respectively, under the action of the discrete symmetry $u_0 \mapsto -u_0$, $\xi \mapsto -\xi$ of (1.4).

Chapter 4

Zero-curvature representations: $\mathbb{Z}_2$-graded case

Let us recall first the definition of Lie super-algebra [10, 86, 92]. Let $\mathcal{A}$ be an algebra over the field $\mathbb{C}$ and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ be the group of residues modulo 2. The algebra $\mathcal{A}$ is called a super-algebra if $\mathcal{A}$ can be decomposed as the direct sum $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ such that

$$\mathcal{A}_{\bar{0}} \cdot \mathcal{A}_{\bar{0}} \subset \mathcal{A}_{\bar{0}}, \quad \mathcal{A}_{\bar{0}} \cdot \mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{1}}, \quad \mathcal{A}_{\bar{1}} \cdot \mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{0}}.$$

A nonzero element of $\mathcal{A}_{\bar{0}}$ or $\mathcal{A}_{\bar{1}}$ is called *homogeneous* (respectively, even or odd). Let $\mathsf{p}(a) = k$ if $a \in \mathcal{A}_k$ for $k \in \mathbb{Z}_2$. The number $\mathsf{p}(a)$ is the parity of a.

The super-algebra $\mathfrak{g}$ is a *Lie super-algebra* if it is endowed with the linear multiplication $\underline{[}\cdot\,,\cdot\underline{]}$ that satisfies the equalities

$$\underline{[}x, y\underline{]} = -(-1)^{\mathsf{p}(x)\mathsf{p}(y)}\underline{[}y, x\underline{]}, \tag{4.1}$$

$$\underline{[}x, \underline{[}y, z\underline{]}\underline{]} = \underline{[}\underline{[}x, y\underline{]}, z\underline{]} + (-1)^{\mathsf{p}(x)\mathsf{p}(y)}\underline{[}y, \underline{[}x, z\underline{]}\underline{]}. \tag{4.2}$$

here x, y, and z are arbitrary elements of $\mathcal{A}$ and x, y are presumed homogeneous.

Let us introduce the super-matrix space $\mathrm{Mat}(p \mid q; \mathcal{A})$. Consider a square $(p+q)$-dimensional matrix $X = \left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right) \in \mathrm{Mat}(p \mid q; \mathcal{A})$ and set

$$\begin{aligned} \mathsf{p}(X) = \bar{0} \quad &\text{if } \mathsf{p}(R_{ij}) = \mathsf{p}(U_{ij}) = \bar{0}, \quad \mathsf{p}(T_{ij}) = \mathsf{p}(S_{ij}) = \bar{1}; \\ \mathsf{p}(X) = \bar{1} \quad &\text{if } \mathsf{p}(R_{ij}) = \mathsf{p}(U_{ij}) = \bar{1}, \quad \mathsf{p}(T_{ij}) = \mathsf{p}(S_{ij}) = \bar{0}. \end{aligned}$$

Taking into account the graded skew-symmetry (4.1) of the bracket $\underline{[}\cdot, \cdot\underline{]}$, we define the Lie super-algebra structure on the space $\mathrm{Mat}(p \mid q; \mathcal{A})$ by the formula

$$\underline{[}X, Y\underline{]} = XY - (-1)^{\mathsf{p}(X)\mathsf{p}(Y)} YX, \quad X, Y \in \mathrm{Mat}(p \mid q; \mathcal{A}). \tag{4.3}$$

The Lie super-algebras $\mathfrak{gl}(m \mid n) \simeq \mathrm{Mat}(m \mid n, \mathbb{C})$ and $\mathfrak{sl}(m \mid n) = \{X \in \mathfrak{gl}(m \mid n) | \operatorname{str} X = 0\}$, where $\operatorname{str}\left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right) = \operatorname{tr} R - \operatorname{tr} U$, are called the general linear and special linear Lie super-algebras, respectively.

To calculate the super-commutator $\underline{[}X, Y\underline{]}$ of two nonhomogeneous elements X and Y, we first split $X = X_{\bar{0}} + X_{\bar{1}}$ and $Y = Y_{\bar{0}} + Y_{\bar{1}}$ so that $\mathsf{p}(X_{\bar{0}}) = \mathsf{p}(Y_{\bar{0}}) = \bar{0}$ and $\mathsf{p}(X_{\bar{1}}) =$

$\mathsf{p}(Y_{\bar{1}}) = \bar{1}$. Using (4.3), we obtain

$$[X, Y] = [X_{\bar{0}} + X_{\bar{1}}, Y_{\bar{0}} + Y_{\bar{1}}] = [X_{\bar{0}}, X_{\bar{0}}] + [X_{\bar{0}}, Y_{\bar{1}}] + [X_{\bar{1}}, Y_{\bar{0}}] + [X_{\bar{1}}, Y_{\bar{1}}] = \\ = (X_{\bar{0}}Y_{\bar{0}} - Y_{\bar{0}}X_{\bar{0}}) + (X_{\bar{0}}Y_{\bar{1}} - Y_{\bar{1}}X_{\bar{0}}) + (X_{\bar{1}}Y_{\bar{0}} - Y_{\bar{0}}X_{\bar{1}}) + (X_{\bar{1}}Y_{\bar{1}} + Y_{\bar{1}}X_{\bar{1}}). \quad (4.4)$$

The super-determinant, or the *Berezinian* of an invertible matrix $X = \left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right) \in \mathfrak{gl}(m \mid n)$ is given by the formula [10]

$$\operatorname{sdet} \begin{pmatrix} R & S \\ T & U \end{pmatrix} = \frac{\det(R - SU^{-1}T)}{\det U}.$$

Example 8. In what follows, we shall use the Lie super-algebra $\mathfrak{sl}(1 \mid 2) \simeq \mathfrak{sl}(2 \mid 1)$, see [41]. Its representation in the space $\mathrm{Mat}(2 \mid 1; \mathbb{C})$ is given by the eight basic vectors, four even: E^+, E^-, H, and Z, and four odd: F^+, F^-, $\bar{F}^+$, and $\bar{F}^-$, where

$$E^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$F^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad F^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \bar{F}^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \bar{F}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The elements of this basis satisfy the following commutation relations:

$$\begin{aligned} &[H, E^\pm] = \pm E^\pm && [H, F^\pm] = \pm\tfrac{1}{2}F^\pm && [H, \bar{F}^\pm] = \pm\tfrac{1}{2}\bar{F}^\pm \\ &[Z, H] = [Z, E^\pm] = 0 && [Z, F^\pm] = \tfrac{1}{2}F^\pm && [Z, \bar{F}^\pm] = -\tfrac{1}{2}\bar{F}^\pm \\ &[E^\pm, F^\pm] = [E^\pm, \bar{F}^\pm] = 0 && [E^\pm, F^\mp] = -F^\pm && [E^\pm, \bar{F}^\mp] = \bar{F}^\pm \\ &[F^\pm, F^\pm] = [\bar{F}^\pm, \bar{F}^\pm] = 0 && [F^\pm, F^\mp] = [\bar{F}^\pm, \bar{F}^\mp] = 0 && [F^\pm, \bar{F}^\pm] = E^\pm \\ &[E^+, E^-] = 2H && [F^\pm, \bar{F}^\mp] = Z \mp H. \end{aligned}$$

The Lie super-algebra $\mathfrak{sl}(2 \mid 1)$ contains the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ as a subalgebra. The vectors $E^\pm$ and H form a basis in $\mathfrak{sl}(2, \mathbb{C})$.

The Lie super-group $SL(2 \mid 1)$, which corresponds to the Lie super-algebra $\mathfrak{sl}(2 \mid 1)$, consist of the matrices with unit Berezinian: $SL(2 \mid 1) = \{S \in GL(2 \mid 1) \mid \operatorname{sdet} S = 1\}$.

Remark 5. Consider the following three subgroups of the Lie super-group $SL(2 \mid 1)$:

$$G_+ = \left\{ \begin{pmatrix} \mathbf{1} & B \\ 0 & 1 \end{pmatrix} \right\}, \quad G_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad G_- = \left\{ \begin{pmatrix} \mathbf{1} & 0 \\ C & 1 \end{pmatrix} \right\}.$$

Each matrix $S \in SL(2 \mid 1)$ can be represented [87] as a product $S = S_+ S_0 S_-$, where $S_+ \in G_+$, $S_0 \in G_0$, $S_- \in G_-$. Due to the multiplicativity of the Berezinian, $\operatorname{sdet} S =$

sdet $S_+ \cdot$ sdet $S_0 \cdot$ sdet $S_- = 1$, and in view of the obvious property sdet $S_+ =$ sdet $S_- = 1$ for all elements of the groups G_+ and G_-, we conclude that sdet $S_0 = 1$ for all $S_0 \in G_0$.

For the Lie super-group $SL(2 \mid 1)$, the dimension of the matrix D is equal to 1×1 and the dimension of the matrix A is equal to 2×2. Let us show that $G_0 \simeq GL(2 \mid 0)$. The condition sdet $S_0 = 1$ for the matrix $S_0 \in SL(2 \mid 1)$ implies the equality $\det A = \det D$ of the usual determinants of A and D. Therefore, to each matrix $A \in GL(2 \mid 0)$ we can put into correspondence the matrix $S_A \in G_0$ by setting $S_A = \left(\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & \det A \end{smallmatrix}\right)$ and conversely, to each matrix $S = \left(\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{smallmatrix}\right) \in G_0$ we associate the matrix A from $GL(2 \mid 0)$.

Consider the tensor product $\mathfrak{g} \otimes_{\mathbb{R}} \bar{\Lambda}(\mathcal{E}^\infty)$ of a finite-dimensional matrix complex Lie superalgebra $\mathfrak{g}$ and the exterior algebra $\bar{\Lambda}(\mathcal{E}^\infty) = \bigoplus_i \Lambda^{0,i}(\mathcal{E}^\infty)$. The product is endowed with the bracket

$$[\![A \otimes \mu, B \otimes \nu]\!] = (-1)^{\mathsf{p}(B)\mathsf{p}(\mu)} [\![A, B]\!] \otimes \mu \wedge \nu$$

for $\mu, \nu \in \bar{\Lambda}(\mathcal{E}^\infty)$ and $A, B \in \mathfrak{g}$. Define the operator $\bar{\mathrm{d}}_h$ that acts on elements of $\mathfrak{g} \otimes \bar{\Lambda}(\mathcal{E}^\infty)$ by the rule

$$\bar{\mathrm{d}}_h(A \otimes \mu) = A \otimes \bar{\mathrm{d}}_h \mu,$$

where the horizontal differential $\bar{\mathrm{d}}_h$ in the right-hand side is (3.1a). The tensor product $\mathfrak{g} \otimes \bar{\Lambda}(\mathcal{E}^\infty)$ is a differential graded associative algebra with respect to the multiplication $(A \otimes \mu) \cdot (B \otimes \nu) = (-1)^{\mathsf{p}(B)\mathsf{p}(\mu)} (A \cdot B) \otimes \mu \wedge \nu$ induced by the ordinary matrix multiplication so that

$$[\![\rho, \sigma]\!] = \rho \cdot \sigma - (-1)^{rs} (-1)^{\mathsf{p}(\rho)\mathsf{p}(\sigma)} \sigma \cdot \rho,$$
$$\bar{\mathrm{d}}_h(\rho \cdot \sigma) = \bar{\mathrm{d}}_h \rho \cdot \sigma + (-1)^r \rho \cdot \bar{\mathrm{d}}_h \sigma$$

for $\rho \in \mathfrak{g} \otimes \bar{\Lambda}^r(\mathcal{E}^\infty)$ and $\sigma \in \mathfrak{g} \otimes \bar{\Lambda}^s(\mathcal{E}^\infty)$. Elements of $\mathfrak{g} \otimes C^\infty(\mathcal{E}^\infty)$ are called $\mathfrak{g}$-*(super)matrices* [94].

Definition 2 ([92, 94, 97])**.** A horizontal 1-form $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E}^\infty)$ is called a $\mathfrak{g}$-valued *zero-curvature representation* for the equation $\mathcal{E}$ if the Maurer–Cartan condition

$$\bar{\mathrm{d}}_h \alpha = \tfrac{1}{2} [\![\alpha, \alpha]\!]. \tag{4.5}$$

holds by virtue of $\mathcal{E}$ and its differential consequences.

Let G be the Lie supergroup of the (matrix) Lie superalgebra $\mathfrak{g}$. Let α and α' be $\mathfrak{g}$-valued zero-curvature representations. Then α and α' are called *guage-equivalent* if there exists $S \in C^\infty(\mathcal{E}^\infty, G)$ such that

$$\alpha' = \alpha^S = \bar{\mathrm{d}}_h S \cdot S^{-1} + S \cdot \alpha \cdot S^{-1}, \qquad . \tag{4.6}$$

Elements of $C^\infty(\mathcal{E}^\infty, G)$, i.e., G-valued functions on $\mathcal{E}^\infty$, are called *G-matrices.*

Definition 3. Let α_λ be a family of zero-curvature representations depending on a complex parameter $\lambda \in \mathcal{I} \subseteq \mathbb{C}$. The parameter λ is *removable* if the forms α_λ are gauge-equivalent at different values of $\lambda \in \mathcal{I}$, and λ is *non-removable* otherwise.

Remark 6. There are other approaches to the idea of parameters' (non)removability, e.g., under transformations which not necessarily are gauge (this is in contrast to the above definition). It turns out that a given parameter in a family of zero-curvature representations can be nonremovable with respect to a narrow class of gauge transformations but, at the same time, it can be eliminated by using transformations from a wider group. For example, Sasaki showed in [114] that the parameter in the standard Lax pair for the Korteweg–de Vries equation can be eliminated by using the scaling symmetry of KdV (see Section 5.5). We stress that this transformation is not gauge and therefore it acts across the gauge group's orbits. However, that parameter is non-removable in the sense of Definition 3 because there is no gauge transformation which would remove it.

We now recall classical Marvan's result and its proof [94, 97] for non-graded PDE and zero-curvature representations (c.f. Proposition 5 below).

Proposition 4 ([94])**.** Let α_λ be a family of $\mathfrak{g}$-valued zero-curvature representations smoothly depending on a complex parameter $\lambda \in \mathcal{I} \subseteq \mathbb{C}$. The parameter λ is removable if and only if for each $\lambda \in \mathcal{I}$ there is a $\mathfrak{g}$-matrix Q_λ, depending smoothly on λ, such that

$$\frac{\partial}{\partial\lambda}\alpha_\lambda = \bar{\mathrm{d}}_h Q_\lambda - [\alpha_\lambda, Q_\lambda].$$

Proof. Suppose that λ is removable. This means that for any fixed λ_0 there exists a G-matrix S_λ such that $\alpha_{\lambda_0}^{S_\lambda} = \alpha_\lambda$ and $S_{\lambda_0} = \mathbf{1} \in G \hookrightarrow C^\infty(\mathcal{E}^\infty, G)$. The matrix $\dot{S}_{\lambda_0} = \partial/\partial\lambda|_{\lambda=\lambda_0} S_\lambda$ belongs to the tangent space at unit of G, i.e., to the Lie algebra $\mathfrak{g}$. We have that

$$\begin{aligned}
0 &= \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \alpha_{\lambda_0} = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \alpha_\lambda^{S_\lambda^{-1}} = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \left(\bar{\mathrm{d}}_h(S_\lambda^{-1})S_\lambda + S_\lambda^{-1}\alpha_\lambda S_\lambda\right) = \\
&= \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \left(-S_\lambda^{-1}\bar{\mathrm{d}}_h S_\lambda + S_\lambda^{-1}\alpha_\lambda S_\lambda\right) = \\
&= -\frac{\partial}{\partial\lambda}\left(S_{\lambda_0}^{-1}\right)\bar{\mathrm{d}}_h(S_{\lambda_0}) - S_{\lambda_0}^{-1} d\dot{S}_{\lambda_0} - S_{\lambda_0}^{-1}\dot{S}_{\lambda_0} S_{\lambda_0}^{-1}\alpha_{\lambda_0} S_{\lambda_0} \\
&\qquad + S_{\lambda_0}^{-1}\dot{\alpha}_{\lambda_0} S_{\lambda_0} + S_{\lambda_0}^{-1}\alpha_{\lambda_0}\dot{S}_{\lambda_0} = -\bar{\mathrm{d}}_h\dot{S}_{\lambda_0} - \dot{S}_{\lambda_0}\alpha_{\lambda_0} + \alpha_{\lambda_0}\dot{S}_{\lambda_0} + \dot{\alpha}_{\lambda_0}.
\end{aligned}$$

This implies that $\dot{\alpha}_{\lambda_0} = \bar{\mathrm{d}}_h\dot{S}_{\lambda_0} - [\alpha_{\lambda_0}, \dot{S}_{\lambda_0}]$, where $\dot{S} \in \mathfrak{g} \otimes \bar{\Lambda}^0(\mathcal{E}^\infty)$.

Conversely, suppose now that $\dot{\alpha}_\lambda = \bar{\mathrm{d}}_h Q_\lambda - [\alpha_\lambda, Q_\lambda]$ for some $Q_\lambda \in \mathfrak{g} \otimes C^\infty(\mathcal{E}^\infty)$. Let $S_\lambda \in C^\infty(\mathcal{E}^\infty, G)$ be a solution of the matrix equation $\partial S/\partial\lambda = Q_\lambda S_\lambda$ with initial data $S_{\lambda_0} = \mathbf{1}$. Consider the expression $Z_\lambda = \bar{\mathrm{d}}_h S_\lambda + S_\lambda\alpha_{\lambda_0} - \alpha_\lambda S_\lambda = (\alpha_{\lambda_0}^{S_\lambda} - \alpha_\lambda)S_\lambda$. We have that

$$\begin{aligned}
\frac{\partial}{\partial\lambda} Z_\lambda &= \frac{\partial}{\partial\lambda}(\bar{\mathrm{d}}_h S_\lambda + S_\lambda \alpha_{\lambda_0} - \alpha_\lambda S_\lambda) \\
&= \bar{\mathrm{d}}_h(\dot{S}_\lambda) + \dot{S}_\lambda \alpha_{\lambda_0} - \dot{\alpha}_\lambda S_\lambda - \alpha_\lambda \dot{S}_\lambda \\
&= \bar{\mathrm{d}}_h(Q_\lambda S_\lambda) + Q_\lambda S_\lambda \alpha_{\lambda_0} - \dot{\alpha}_\lambda S_\lambda - \alpha_\lambda Q_\lambda S_\lambda = \\
&= \bar{\mathrm{d}}_h Q_\lambda S_\lambda + Q_\lambda \bar{\mathrm{d}}_h S_\lambda + Q_\lambda S_\lambda \alpha_{\lambda_0} - \dot{\alpha}_\lambda S_\lambda - \alpha_\lambda Q_\lambda S_\lambda + (Q_\lambda \alpha_\lambda S_\lambda - Q_\lambda \alpha_\lambda S_\lambda) \\
&= (\underbrace{\bar{\mathrm{d}}_h Q_\lambda - \alpha_\lambda Q_\lambda + Q_\lambda \alpha_\lambda}_{\dot{\alpha}_\lambda} - \dot{\alpha}_\lambda) S_\lambda + Q_\lambda(\bar{\mathrm{d}}_h S_\lambda + S_\lambda \alpha_{\lambda_0} - \alpha_\lambda S_\lambda) \\
&= Q_\lambda Z_\lambda.
\end{aligned}$$

It is obvious that $Z_{\lambda_0} = 0$, whence $\alpha_{\lambda_0}^{S_\lambda} - \alpha_\lambda = 0$. Therefore, the parameter λ is removable. $\square$

The following proposition and its proof are proper $\mathbb{Z}_2$-generalizations of Marvan's Proposition 4 for classical, non-graded systems of partial differential equations [94, 97].

Proposition 5 ([66])**.** Let α_λ be a family of $\mathfrak{g}$-valued zero-curvature representations smoothly depending on a complex parameter $\lambda \in \mathcal{I} \subseteq \mathbb{C}$. The parameter λ is removable if and only if for each $\lambda \in \mathcal{I}$ there is a $\mathfrak{g}$-matrix Q_λ, depending smoothly on λ, such that $\mathsf{p}(Q_\lambda) = \bar{0}$ and

$$\frac{\partial}{\partial\lambda}\alpha_\lambda = \bar{\mathrm{d}}_h Q_\lambda - [\![\alpha_\lambda, Q_\lambda]\!].$$

Proof. The first half of the proof (i.e., the necessity) coincides literaly with the proof of Proposition 4; note that party of $\dot{S}_\lambda$ will be the same as parity S_λ, i.e. $\mathsf{p}(\dot{S}_{\lambda_0}) = \bar{0}$. This means that for any fixed λ_0 there exists a G-matrix S_λ such that $\alpha_{\lambda_0}^{S_\lambda} = \alpha_\lambda$ and $S_{\lambda_0} = \mathbf{1} \in G \hookrightarrow C^\infty(\mathcal{E}^\infty, G)$. The matrix $\dot{S}_{\lambda_0} = \partial/\partial\lambda|_{\lambda=\lambda_0} S_\lambda$ belongs to the tangent space at unit of G, i.e., to the super Lie superalgebra $\mathfrak{g}$.

The converse s true by the same argument as above; we note that solutions S_λ exists only for even $\mathfrak{g}$-matrices Q_λ. $\square$

Remark 7. We conclude that Marvan's computational techniques [94, 97] work also in the $\mathbb{Z}_2$-graded setup — with just one modification: the commutator $[\cdot,\cdot]$ in a Lie algebra is replaced by the graded commutator $[\![\cdot,\cdot]\!]$ in the Lie superalgebra. However, let us say a word of caution.

Lemma 1 ([66])**.** Let $\alpha = \alpha^{\bar{0}} + \alpha^{\bar{1}}$ be a $\mathfrak{g}$-valued zero-curvature representation of a given $\mathbb{Z}_2$-graded equation $\mathcal{E}$ such that $\mathsf{p}(\alpha^{\bar{0}}) = \bar{0}$ and $\mathsf{p}(\alpha^{\bar{1}}) = \bar{1}$. Then Marvan's operator $\bar{\boldsymbol{\partial}}_\alpha = \bar{\mathrm{d}}_h - [\![\alpha, \cdot]\!]$, see [94], not necessarily is a differential.

We note that we have not seen any example of a ZCR with nonzero odd part (i.e., such that $\alpha^{\bar{1}} \neq 0$). It would be interesting to either find such example or prove that it can not exist.

Proof. Let $\beta \in \mathfrak{g} \otimes \bar{\Lambda}^0(\mathcal{E}^\infty)$ so that $\beta = \beta^{\bar{0}} + \beta^{\bar{1}}$ and consider $\alpha = \alpha^{\bar{0}} + \alpha^{\bar{1}}$, where $\mathsf{p}(\alpha^{\bar{0}}) = \mathsf{p}(\beta^{\bar{0}}) = \bar{0}$ and $\mathsf{p}(\alpha^{\bar{1}}) = \mathsf{p}(\beta^{\bar{1}}) = \bar{1}$. Then we have that

$$
\begin{aligned}
&\bar{\boldsymbol{\partial}}_\alpha \circ \bar{\boldsymbol{\partial}}_\alpha(\beta) = \bar{\boldsymbol{\partial}}_\alpha(\bar{\mathrm{d}}_h\beta - [\alpha,\beta]) = \bar{\mathrm{d}}_h \circ \bar{\mathrm{d}}_h\beta - \bar{\mathrm{d}}_h([\alpha,\beta]) - [\alpha, \bar{\mathrm{d}}_h\beta - [\alpha,\beta]]\\
&\quad = -[\bar{\mathrm{d}}_h\alpha,\beta] + [\alpha,\bar{\mathrm{d}}_h\beta] - [\alpha,\bar{\mathrm{d}}_h\beta] + [\alpha,[\alpha,\beta]] = [\alpha,[\alpha,\beta]] - \tfrac{1}{2}[[\alpha,\alpha],\beta]\\
&\quad = [\alpha, \alpha^{\bar{0}}\beta^{\bar{0}} - \beta^{\bar{0}}\alpha^{\bar{0}} + \alpha^{\bar{0}}\beta^{\bar{1}} - \beta^{\bar{1}}\alpha^{\bar{0}} + \alpha^{\bar{1}}\beta^{\bar{0}} - \beta^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\beta^{\bar{1}} + \beta^{\bar{1}}\alpha^{\bar{1}}] - [\alpha^{\bar{0}}\alpha^{\bar{0}} + \alpha^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{0}}, \beta]\\
&\quad = \alpha^{\bar{0}}\alpha^{\bar{0}}\beta^{\bar{0}} + \alpha^{\bar{0}}\beta^{\bar{0}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\beta^{\bar{0}}\alpha^{\bar{0}} - \beta^{\bar{0}}\alpha^{\bar{0}}\alpha^{\bar{0}} + \alpha^{\bar{0}}\alpha^{\bar{0}}\beta^{\bar{1}} + \alpha^{\bar{0}}\beta^{\bar{1}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\beta^{\bar{1}}\alpha^{\bar{0}} - \beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{0}}\\
&\qquad + \alpha^{\bar{0}}\alpha^{\bar{1}}\beta^{\bar{0}} + \alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\beta^{\bar{0}}\alpha^{\bar{1}} - \beta^{\bar{0}}\alpha^{\bar{1}}\alpha^{\bar{0}} + \alpha^{\bar{0}}\alpha^{\bar{1}}\beta^{\bar{1}} + \alpha^{\bar{1}}\beta^{\bar{1}}\alpha^{\bar{0}} + \alpha^{\bar{1}}\alpha^{\bar{0}}\beta^{\bar{0}} + \alpha^{\bar{0}}\beta^{\bar{0}}\alpha^{\bar{1}}\\
&\qquad - \alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{0}} - \beta^{\bar{0}}\alpha^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{0}}\beta^{\bar{1}} - \alpha^{\bar{0}}\beta^{\bar{1}}\alpha^{\bar{1}} - \alpha^{\bar{1}}\beta^{\bar{1}}\alpha^{\bar{0}} - \beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{1}}\beta^{\bar{0}} - \alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{1}}\\
&\qquad - \alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{1}} + \beta^{\bar{0}}\alpha^{\bar{1}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{1}}\beta^{\bar{1}} + \alpha^{\bar{1}}\beta^{\bar{1}}\alpha^{\bar{1}} - \alpha^{\bar{0}}\alpha^{\bar{0}}\beta^{\bar{0}} + \beta^{\bar{0}}\alpha^{\bar{0}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\alpha^{\bar{0}}\beta^{\bar{1}} + \beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{0}}\\
&\qquad - \alpha^{\bar{1}}\alpha^{\bar{0}}\beta^{\bar{0}} + \beta^{\bar{0}}\alpha^{\bar{1}}\alpha^{\bar{0}} - \alpha^{\bar{1}}\alpha^{\bar{0}}\beta^{\bar{1}} - \beta^{\bar{1}}\alpha^{\bar{1}}\alpha^{\bar{0}} - \alpha^{\bar{0}}\alpha^{\bar{1}}\beta^{\bar{0}} + \beta^{\bar{0}}\alpha^{\bar{0}}\alpha^{\bar{1}} - \alpha^{\bar{0}}\alpha^{\bar{1}}\beta^{\bar{1}} - \beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{1}}\\
&\quad = -\alpha^{\bar{0}}\beta^{\bar{1}}\alpha^{\bar{1}} - 2\beta^{\bar{1}}\alpha^{\bar{0}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{1}}\beta^{\bar{0}} - 2\alpha^{\bar{1}}\beta^{\bar{0}}\alpha^{\bar{1}} + \beta^{\bar{0}}\alpha^{\bar{1}}\alpha^{\bar{1}} + \alpha^{\bar{1}}\alpha^{\bar{1}}\beta^{\bar{1}} + \alpha^{\bar{1}}\beta^{\bar{1}}\alpha^{\bar{1}} - \beta^{\bar{1}}\alpha^{\bar{1}}\alpha^{\bar{0}} \neq 0.
\end{aligned}
$$

This argument shows that for *parity-even* zero-curvature representations (which are constrained by $\alpha^{\bar{1}} = 0$) the operator $\bar{\boldsymbol{\partial}}_\alpha$ is a differential, and Marvan's cohomology technique [94] works also in the $\mathbb{Z}_2$-graded setup. ☐

Example 9. Let us consider the four-component generalization of the KdV equation, namely, the N=2 supersymmetric Korteweg–de Vries equation (SKdV) [84]:

$$
\boldsymbol{u}_t = -\boldsymbol{u}_{xxx} + 3\big(\boldsymbol{u}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}\big)_x + \frac{a-1}{2}\big(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}^2\big)_x + 3a\boldsymbol{u}^2\boldsymbol{u}_x, \qquad \mathcal{D}_i = \frac{\partial}{\partial\theta_i} + \theta_i \cdot \bar{D}_x, \tag{1.2}
$$

where

$$
\boldsymbol{u}(x,t;\theta_1,\theta_2) = u_0(x,t) + \theta_1 \cdot u_1(x,t) + \theta_2 \cdot u_2(x,t) + \theta_1\theta_2 \cdot u_{12}(x,t) \tag{3.2}
$$

is the complex bosonic super-field, θ_1, θ_2 are Grassmann variables such that $\theta_1^2 = \theta_2^2 = \theta_1\theta_2 + \theta_2\theta_1 = 0$, u_0, u_{12} are bosonic fields ($\mathsf{p}(u_0) = \mathsf{p}(u_{12}) = \bar{0}$), and u_1, u_2 are fermionic fields ($\mathsf{p}(u_1) = \mathsf{p}(u_2) = \bar{1}$). Expansion (3.2) converts (1.2) to the four-component system[1] (3.5)

$$
\begin{aligned}
u_{0;t} &= -u_{0;xxx} + \big(au_0^3 - (a+2)u_0u_{12} + (a-1)u_1u_2\big)_x,\\
u_{1;t} &= -u_{1;xxx} + \big(\quad (a+2)u_0u_{2;x} + (a-1)u_{0;x}u_2 - 3u_1u_{12} + 3au_0^2u_1\big)_x,\\
u_{2;t} &= -u_{2;xxx} + \big(-(a+2)u_0u_{1;x} - (a-1)u_{0;x}u_1 - 3u_2u_{12} + 3au_0^2u_2\big)_x,\\
\underline{u_{12;t}} &= \underline{-u_{12;xxx} - 6u_{12}u_{12;x}} + 3au_{0;x}u_{0;xx} + (a+2)u_0u_{0;xxx}\\
&\qquad + 3u_1u_{1;xx} + 3u_2u_{2;xx} + 3a\big(u_0^2u_{12} - 2u_0u_1u_2\big)_x.
\end{aligned}
$$

[1]The Korteweg–de Vries equation upon u_{12}, see (1.1), is underlined.

The N=2 supersymmetric a=4-KdV equation (3.5) admits [28] the $\mathfrak{sl}(2|1)$-valued zero-curvature representation $\alpha^{N=2} = A\,\mathrm{d}x + B\,\mathrm{d}t$, where[2]

$$A = \begin{pmatrix} -\boldsymbol{i}u_0 & \varepsilon^{-1}(u_0^2+u_{12}) - \varepsilon^{-2}u_0\boldsymbol{i} & -\varepsilon^{-1}(u_2+\boldsymbol{i}u_1) \\ -\varepsilon & -\boldsymbol{i}u_0 - \varepsilon^{-1} & 0 \\ 0 & \boldsymbol{i}u_1 - u_2 & -2\boldsymbol{i}u_0 - \varepsilon^{-1} \end{pmatrix};$$

The elements of the $\mathfrak{sl}(2|1)$-matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

are as follows,

$$\begin{aligned}
b_{11} &= 4\boldsymbol{i}u_0^3 - 6\boldsymbol{i}u_0u_{12} + 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} - u_{12;x} - 4\boldsymbol{i}u_2u_1 + \varepsilon^{-1}(2u_0^2 - u_{12} - \boldsymbol{i}u_{0;x}) - \boldsymbol{i}\varepsilon^{-2}u_0,\\
b_{12} &= \varepsilon^{-1}(4u_0^4 + 2u_0^2u_{12} + 4u_0u_{0;xx} - 2u_{12}^2 + 4u_{0;x}^2 - u_{12;xx} + u_2u_{2;x} + 8u_2u_1u_0 + u_1u_{1;x}) +\\
&\quad + \varepsilon^{-2}(2\boldsymbol{i}u_0^3 - 4\boldsymbol{i}u_0u_{12} + 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} - u_{12;x} - 2\boldsymbol{i}u_2u_1) + \varepsilon^{-3}(u_0^2 - u_{12} - \boldsymbol{i}u_{0;x}) -\\
&\quad - \boldsymbol{i}\varepsilon^{-4}u_0,\\
b_{13} &= \varepsilon^{-1}(-5\boldsymbol{i}u_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} + \boldsymbol{i}u_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} - 4\boldsymbol{i}u_2u_{0;x} - 8\boldsymbol{i}u_1u_0^2 +\\
&\quad + 2\boldsymbol{i}u_1u_{12} - 4u_1u_{0;x}) + \varepsilon^{-2}(-u_{2;x} + \boldsymbol{i}u_{1;x} - 3\boldsymbol{i}u_2u_0 - 3u_1u_0) + \varepsilon^{-3}(-u_2 + \boldsymbol{i}u_1),\\
b_{21} &= 2\varepsilon(-2u_0^2 + u_{12}) + 2\boldsymbol{i}u_0 + \varepsilon^{-1},\\
b_{22} &= 4\boldsymbol{i}u_0^3 - 6\boldsymbol{i}u_0u_{12} - 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} + u_{12;x} - 4\boldsymbol{i}u_2u_1 + \varepsilon^{-1}(-2u_0^2 + u_{12} + \boldsymbol{i}u_{0;x}) +\\
&\quad + \boldsymbol{i}\varepsilon^{-1}u_0 + \varepsilon^{-3},\\
b_{23} &= u_{2;x} - \boldsymbol{i}u_{1;x} + 4\boldsymbol{i}u_2u_0 + 4u_1u_0 + \varepsilon^{-1}(u_2 - \boldsymbol{i}u_1),\\
b_{31} &= \varepsilon(-u_{2;x} - \boldsymbol{i}u_{1;x} + 4\boldsymbol{i}u_2u_0 - 4u_1u_0) + u_2 + \boldsymbol{i}u_1,\\
b_{32} &= 5\boldsymbol{i}u_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} - \boldsymbol{i}u_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} + 4\boldsymbol{i}u_2u_{0;x} + 8\boldsymbol{i}u_1u_0^2 - 2\boldsymbol{i}u_1u_{12} -\\
&\quad - 4u_1u_{0;x} + \varepsilon^{-1}u_0(\boldsymbol{i}u_2 - u_1),\\
b_{33} &= 2(4\boldsymbol{i}u_0^3 - 6\boldsymbol{i}u_0u_{12} - \boldsymbol{i}u_{0;xx} - 4\boldsymbol{i}u_2u_1) + \varepsilon^{-3}.
\end{aligned}$$

[2]This zero-curvature representation is not equal identically but it is gauge-equivalent to the respective formula in Das *et al.* [28]. The transformation between these objects contains the imaginary unit $\boldsymbol{i}$. Our choice of normalization is due to the following argument: all structures under study contain Gardner's deformation (5.41) of Korteweg–de Vries equation (1.1) (so that the structures retract to Gardner's deformation under suitable reductions).

We note further that the zero-curvature representation $\alpha^{N=2}$ can be used for construction of a solution, which is an alternative to the first solution reported in Chapter 3, of Gardner's deformation problem [84, 99] for the N=2, a=4 SKdV equation (we refer to Chapter 5 for detail). The parameter ε which we use here is the parameter in the classical Gardner deformation of the KdV equation [101]. Therefore, we denote this parameter by ε instead of λ.

We claim that there is no $\mathfrak{sl}(2|1)$-matrix Q satisfying the equalities

$$\frac{\partial}{\partial\varepsilon}A = \bar{D}_x(Q) - [\underline{A}, Q\underline{]}, \qquad \frac{\partial}{\partial\varepsilon}B = \bar{D}_t(Q) - [\underline{B}, Q\underline{]}.$$

Consequently, the parameter ε in $\alpha^{N=2}$ is non-removable under gauge transformations.

Example 10. Consider another $\mathfrak{sl}(2|2)$-valued zero-curvature representation $\beta = A\,\mathrm{d}x + B\,\mathrm{d}t$ for the N=2, a=4-SKdV equation: we let

$$A = \begin{pmatrix} \lambda - \boldsymbol{i}u_0 & -\lambda^2 - (u_0^2 + u_{12}) & -\boldsymbol{i}u_1 - u_2 \\ 1 & -\lambda - \boldsymbol{i}u_0 & 0 \\ 0 & u_2 - \boldsymbol{i}u_1 & -2\boldsymbol{i}u_0 \end{pmatrix};$$

The elements of the $\mathfrak{sl}(2|1)$-matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

are given by the formulas

$$\begin{aligned}
b_{11} &= 2\lambda(2u_0^2 - u_{12}) - 4\boldsymbol{i}u_0^3 + 6\boldsymbol{i}u_0u_{12} + 4u_0u_{0;x} + \boldsymbol{i}u_{0;xx} - u_{12;x} + 4\boldsymbol{i}u_2u_1, \\
b_{12} &= 2\lambda^2(-2u_0^2 + u_{12}) + 2\lambda(-4u_0u_{0;x} + u_{12;x}) - 4u_0^4 - 2u_0^2u_{12} - 4u_0u_{0;xx} + 2u_{12}^2 \\
&\quad - 4u_{0;x}^2 + u_{12;xx} - u_2u_{2;x} - 8u_2u_1u_0 - u_1u_{1;x}, \\
b_{13} &= \lambda(u_{2;x} + \boldsymbol{i}u_{1;x} - 4\boldsymbol{i}u_2u_0 + 4u_1u_0) - 5\boldsymbol{i}u_0u_{2;x} + 5u_0u_{1;x} + u_{2;xx} + \boldsymbol{i}u_{1;xx} \\
&\quad - 8u_2u_0^2 + 2u_2u_{12} - 4\boldsymbol{i}u_2u_{0;x} - 8\boldsymbol{i}u_1u_0^2 + 2\boldsymbol{i}u_1u_{12} + 4u_1u_{0;x}, \\
b_{21} &= 2(2u_0^2 - u_{12}), \\
b_{22} &= 2\lambda(-2u_0^2 + u_{12}) - 4\boldsymbol{i}u_0^3 + 6\boldsymbol{i}u_0u_{12} - 4u_0u_{0;x} + \boldsymbol{i}u_{0;xx} + u_{12;x} + 4\boldsymbol{i}u_2u_1, \\
b_{23} &= u_{2;x} + \boldsymbol{i}u_{1;x} - 4\boldsymbol{i}u_2u_0 + 4u_1u_0, \\
b_{31} &= u_{2;x} - \boldsymbol{i}u_{1;x} + 4\boldsymbol{i}u_2u_0 + 4u_1u_0, \\
b_{32} &= \lambda(-u_{2;x} + \boldsymbol{i}u_{1;x} - 4\boldsymbol{i}u_2u_0 - 4u_1u_0) - 5\boldsymbol{i}u_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} + \boldsymbol{i}u_{1;xx} + 8u_2u_0^2 \\
&\quad - 2u_2u_{12} - 4\boldsymbol{i}u_2u_{0;x} - 8\boldsymbol{i}u_1u_0^2 + 2\boldsymbol{i}u_1u_{12} - 4u_1u_{0;x}, \\
b_{33} &= 2\boldsymbol{i}(-4u_0^3 + 6u_0u_{12} + u_{0;xx} + 4u_2u_1).
\end{aligned}$$

The $\mathfrak{sl}(2|1)$-matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfies the equations

$$\frac{\partial}{\partial\lambda}A = \bar{D}_x(Q) - [\underline{A}, Q\underline{]}, \qquad \frac{\partial}{\partial\lambda}B = \bar{D}_t(Q) - [\underline{B}, Q\underline{]}.$$

Solving the Cauchy problem

$$\frac{\partial}{\partial\lambda}S = QS, \quad S|_{\lambda=0} = \mathbf{1},$$

we obtain the $SL(2|1)$-matrix

$$S = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix S defines the gauge transformation that removes the parameter λ from the zero-curvature representation β, i.e., $(\beta)^{S^{-1}} = \beta|_{\lambda=0}$. Consequently, the parameter λ in β is removable.

We extended – to the $\mathbb{Z}_2$-graded case – Marvan's method for inspecting the (non)removability of a parameter in a given family of zero-curvature representations; specifically, we accomplished the task of balancing the signs in a nonselfcontradictory way. Let us note that this generalization of the standard technique can be used further in solving Gardner's deformation problems for the N=2-supersymmetric KdV equations and other $\mathbb{Z}_2$-graded completely integrable systems (see Chapter 5).

Chapter 5

Non-local picture

The classical problem of construction of the Gardner deformation [101] for an infinite-dimensional completely integrable system of evolutionary partial differential equations essentially amounts to finding a recurrence relation between the integrals of motion. For the N=2 supersymmetric generalizations of the Korteweg–de Vries equation [84, 99], the deformation problem was posed when the integrable triplet of such super-systems was discovered. In Chapter 3 we proved the 'no-go' theorem stating that a classical polynomial Gardner deformation for the N=2 supersymmetric a=4 KdV equation does *not* exist within the superfield formalism (but that in principle, the deformation may exist whenever the superfields are split in components), c.f. [84]. This is in contrast with the N=1 sKdV case when the two approaches yield the supersymmetry-invariant deformation [99].

In this chapter we re-address, from a basically different viewpoint, the Gardner deformation problem for a vast class of (not necessarily supersymmetric) KdV-like systems. Namely, in Chapter 3 we emphasized the geometric similarity of the Gardner deformations and zero-curvature representations, each of them manifesting the integrability of nonlinear systems (c.f. [43, 123]). Indeed, both constructions generate infinite sequences of nontrivial integrals of motion. However, the standard Lax approach relies on the calculus of pseudodifferential operators whereas the Gardner technique is more geometric and favourable from a computational viewpoint.

Developing further the approach of [111], we reformulate the Gardner deformation problem for the graded extensions of the KdV equation in terms of constructing parameter-dependent families of new bosonic and fermionic variables. We require that the 'nonlocalities' possess two defining properties ([59, 78]): on the one hand, they should reproduce the classical Gardner deformation from [101] under the shrinking of the N=2 super-equation back to the KdV equation. On the other hand, we consider the nonlocalities that encode the parameter-dependent zero-curvature representations for the super-systems at hand. In this reformulation, we solve Mathieu's Open problem 2 of [99] for the N=2 supersymmetric a=4-KdV equation. However, our approach is applicable to a much wider class of completely integrable (super-)systems.

In the recent paper [59] Kiselev understood Gardner's deformations in the extended sense, namely, in terms of coverings over PDE and diagrams of coverings. Zero-curvature representations and Gardner's deformations can be considered as such geometric structures[1] that obey some extra conditions.

5.1 Differential coverings and zero-curvature representations

Definition 4 ([12, 80])**.** Let $\mathcal{E}$ be a differential equation that admits the nonempty infinite prolongation $\mathcal{E}^\infty$. A *covering* (or differential covering) over the equation $\mathcal{E}$ is another (usually, larger) system of partial differential equations $\tilde{\mathcal{E}}$ endowed with the n-dimensional Cartan distribution $\tilde{\mathcal{C}}$ and such that there is a mapping $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ for which, at each point $\theta \in \tilde{\mathcal{E}}$ the tangent map $\tau_{*,\theta}$ is an isomorphism of the plane $\tilde{\mathcal{C}}_\theta$ to the Cartan plane $\mathcal{C}_{\tau(\theta)}$ at the point $\tau(\theta)$ in $\mathcal{E}^\infty$.

The construction of a covering over $\mathcal{E}$ means the introduction of new variables such that their compatibility conditions lie inside the initial system $\mathcal{E}^\infty$. In practice (see [62]), it is the rules to differentiate the new variable which are specified in a consistent way; this implies that those new variables acquire the nature of nonlocalities if their derivatives are local but the variables themselves are not (e.g., consider the potential $\mathfrak{v} = \int u\,\mathrm{d}x$ satisfying $\mathfrak{v}_x = u$ and $\mathfrak{v}_t = -u_{xx} - 3u^2$ for the KdV equation $u_t + u_{xxx} + 6uu_x = 0$). Whenever the covering is indeed realized as the fibre bundle $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$, the forgetful map τ discards the nonlocalities.

In these terms, zero-curvature representations and Gardner's deformations are coverings of special kinds (see Examples 12 and 15 below). We use the geometric similarity of the two notions and construct new Gardner's deformations from known zero-curvature representations (but this is *not always* possible[2]).

Example 11 (A zero-curvature representation for the KdV equation)**.** Consider the Korteweg–de Vries (KdV) equation (1.1) and its Lax representation [37, 101, 103]

$$\mathcal{L}_t = [\mathcal{L}, \mathcal{A}],$$

where

$$\mathcal{L} = \tfrac{\mathrm{d}^2}{\mathrm{d}x^2} + u_{12}, \quad \mathcal{A} = -4\tfrac{\mathrm{d}^3}{\mathrm{d}x^3} - 6u_{12}\tfrac{\mathrm{d}}{\mathrm{d}x} - 3u_{12;x}. \tag{5.1}$$

The linear auxiliary problem [128] is

$$\begin{aligned} \psi_{xx} + u_{12}\psi &= \lambda\psi, \\ -4\psi_{xxx} - 6u\psi_x - 3u_{12;x}\psi &= \psi_t, \end{aligned}$$

[1] Bäcklund (auto)transformations between PDE appear in the same context. In [59] Kiselev argued that the former, when regarded as the diagrams, are dual to the diagram description of Gardner's deformations.

[2] For example, Gardner's deformation (2.3)–(2.4) does not correspond to any zero-curvature representation with values in a finite-dimensional Lie algebra.

By definition, put $\psi_0 = \psi$ and $\psi_1 = \psi_x$. We obtain

$$\begin{aligned}
\psi_{0;x} &= \psi_1,\\
\psi_{1;x} &= (\lambda - u_{12})\psi_0,\\
\psi_{0;t} &= -4\tfrac{\mathrm{d}}{\mathrm{d}x}((\lambda - u_{12})\psi_0) - 6u_{12}\psi_1 - 3u_{12;x}\psi_0 = u_{12;x}\psi_0 + (-4\lambda - 2u_{12})\psi_1,\\
\psi_{1;t} &= (-4\lambda^2 + 2u_{12}\lambda + 2u_{12}^2 + u_{12;x})\psi_0 + (-u_{12;x})\psi_1.
\end{aligned}$$

We finally rewrite this system as two matrix equations [128],

$$\underbrace{\begin{pmatrix}\psi_{0;x}\\ \psi_{1;x}\end{pmatrix}}_{\psi_x} = \underbrace{\begin{pmatrix}0 & 1\\ \lambda - u_{12} & 0\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}}_{\psi}$$

$$\underbrace{\begin{pmatrix}\psi_{0;t}\\ \psi_{1;t}\end{pmatrix}}_{\psi_t} = \underbrace{\begin{pmatrix}u_{12;x} & -4\lambda - 2u_{12}\\ -4\lambda^2 + 2u_{12}\lambda + 2u_{12}^2 + u_{12;xx} & -u_{12;x}\end{pmatrix}}_{B}\underbrace{\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}}_{\psi}.$$

This yields an $\mathfrak{sl}_2(\mathbb{C})$-valued zero-curvature representation $\alpha^{\mathrm{KdV}} = A\,\mathrm{d}x + B\,\mathrm{d}t$ for the KdV equation (1.1). The representation α^{KdV} was rediscovered in [93].

Example 12 (Zero-curvature representations as coverings)**.** Let $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ as above. We introduce the standard basis e, h, f in $\mathfrak{g}$ such that

$$[e,h] = -2e, \quad [e,f] = h, \quad [f,h] = 2f.$$

We consider, simultaneously, the matrix representation

$$\rho:\ \mathfrak{sl}_2(\mathbb{C}) \to \{M \in \mathrm{Mat}(2,2) \mid \mathrm{tr}\, M = 0\}$$

of $\mathfrak{g}$ and its representation ϱ in the space of vector fields with polynomial coefficients on the complex line with the coordinate w:

$$\begin{array}{rclrclrcl}
\rho(e) &=& \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, & \rho(h) &=& \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, & \rho(f) &=& \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\\
\varrho(e) &=& 1\cdot\partial/\partial w, & \varrho(h) &=& -2w\cdot\partial/\partial w, & \varrho(f) &=& -w^2\cdot\partial/\partial w.
\end{array}$$

Let us decompose the matrices $A_i \in C^\infty(\mathcal{E}^\infty) \otimes \mathfrak{g}$ (which occur in the zero-curvature representation $\alpha = \sum_i A_i \mathrm{d}x^i$) with respect to the basis in the space $\rho(\mathfrak{g})$,

$$A_i = a_e^{(i)} \otimes \rho(e) + a_h^{(i)} \otimes \rho(h) + a_f^{(i)} \otimes \rho(f), \tag{5.2}$$

for $a_j^{(i)} \in C^\infty(\mathcal{E}^\infty)$.

To construct the covering $\tilde{\mathcal{E}}$ over $\mathcal{E}^\infty$ with a new fiber variable w over $\mathcal{E}^\infty$ (the 'nonlocality'), we switch from the representation ρ to ϱ. We thus obtain the vector fields

$$V_{A_i} = a_e^{(i)} \otimes \varrho(e) + a_h^{(i)} \otimes \varrho(h) + a_f^{(i)} \otimes \varrho(f) \tag{5.2$'$}$$

such that the prolongations of the total derivatives D_{x^i} to $\tilde{\mathcal{E}}$ are defined by the formula

$$\tilde{D}_{x^i} = D_{x^i} - V_{A_i}. \tag{5.3}$$

The extended derivatives act on the nonlocal variable w as follows,

$$\tilde{D}_{x^i} w = \mathrm{d}w \,\lrcorner\, (-V_{A_i}).$$

Remark 8. The commutativity of the prolonged total derivatives, $\big[\tilde{D}_{x^i}, \tilde{D}_{x^j}\big] = 0$ with $i \neq j$, is equivalent to the Maurer–Cartan equation (4.5): Indeed, we have that

$$0 = [\tilde{D}_{x^i}, \tilde{D}_{x^j}] = [D_{x^i} - V_{A_i}, D_{x^j} - V_{A_j}] = [D_{x^i}, D_{x^j}] - [D_{x^i}, V_{A_j}] - [V_{A_j}, D_{x^j}] + [V_{A_i}, V_{A_j}] =$$
$$= -V_{D_{x^i} A_i} + V_{D_{x^j} A_i} + V_{[A_i, A_j]} = V_{D_{x^j} A_i - D_{x^i} A_j + [A_i, A_j]} \ \Leftrightarrow\ D_{x^j} A_i - D_{x^i} A_j + [A_i, A_j] = 0.$$

This motivates the choice of the minus sign in (5.3).

Example 13 (A one-dimensional covering over the KdV equation)**.** One obtains the covering over the KdV equation from the zero-curvature representation α (see Example 11) by using representation (5.2$'$) in the space of vector fields. Applying (5.2$'$) to the matrices A, $B \in \mathfrak{sl}_2(\mathbb{C})$, we construct the following vector fields with the nonlocal variable w:

$$V_A = (1 - (\lambda - u_{12}) w^2) \cdot \partial/\partial w,$$
$$V_B = \big[(-4\lambda - 2u_{12}) - 2u_{12} w - (-4\lambda^2 + 2u_{12}\lambda + 2u_{12}^2 + u_{12;xx}) w^2\big] \cdot \partial/\partial w.$$

The prolongations of the total derivatives act on w by the rules

$$w_x = -1 + (\lambda - u_{12}) w^2, \tag{5.4a}$$
$$w_t = -\big((-4\lambda - 2u_{12}) - 2u_{12;x} w - (-4\lambda^2 + 2u_{12}\lambda + 2u_{12}^2 + u_{12;xx}) w^2\big). \tag{5.4b}$$

We thus obtain the one-dimensional covering over the KdV equation (1.1). In what follows we show that this covering is equivalent to the covering (5.8) which is derived from Gardner's deformation (2.1) of the KdV equation (1.1).

Example 14 (The projective substitution and nonlinear realizations of Lie algebras in the spaces of vector fields [111])**.** Let N be a $(k_0 + 1 | k_1)$-dimensional supermanifold with local coordinates

$$\boldsymbol{v} = (v^1, v^2, \ldots, v^{k_0+1}, f^1, \ldots, f^{k_1}) \in N, \text{ and put } \partial_{\boldsymbol{v}} = (\partial_{v^1}, \partial_{v^2}, \ldots, \partial_{v^{k_0+1}}, \partial_{f^1}, \ldots, \partial_{f^{k_1}})^{\mathrm{t}}.$$

For any $g \in \mathfrak{g} \subseteq \mathfrak{gl}(k_0+1|k_1)$, its image V_g under the representation of $\mathfrak{g}$ in the space of vector fields on N is given by the formula

$$V_g = \boldsymbol{v} g \partial_{\boldsymbol{v}}.$$

We note that V_g is linear in v^i and f^j. By construction, the representation preserves all commutation relations in the initial Lie algebra $\mathfrak{g}$:

$$[\![V_g, V_h]\!] = [\![\boldsymbol{v} g\partial_{\boldsymbol{v}}, \boldsymbol{v} h \partial_{\boldsymbol{v}}]\!] = \boldsymbol{v}[\![g,h]\!]\partial_{\boldsymbol{v}} = V_{[\![g,h]\!]}, \quad h, g \in \mathfrak{g}.$$

Locally, at all points of N where $v_1 \neq 0$ we consider the projection

$$p\colon v^i \mapsto w^{i-1} = \mu v^i / v^1, \quad p\colon f^j_{\text{old}} \mapsto f^j_{\text{new}} = \mu f^j_{\text{old}} / v^1, \qquad \mu \in \mathbb{R}, \tag{5.5}$$

and its differential $\mathrm{d}p\colon \partial_{\boldsymbol{v}} \to \partial_{\boldsymbol{w}}$. The transformation p yields new coordinates on the open subset of N where $v^1 \neq 0$ and determines a basis in the fibres of TN over that subset:

$$\begin{aligned} \boldsymbol{w} &= (\mu, w^1, \ldots, w^{k_0}, f^1, \ldots, f^{k_1}), \\ \partial_{\boldsymbol{w}} &= \Bigl(-\frac{1}{\mu}\Bigl(\sum_{i=1}^{k_0} w^i \partial_{w^i} + \sum_{j=1}^{k_1} f^j \partial_{f^j}\Bigr), \partial_{w^2}, \ldots, \partial_{w^{k_0}}, \partial_{f^1}, \ldots, \partial_{f^{k_1}}\Bigr)^{\mathrm{t}}. \end{aligned}$$

Consider the vector field $X_g = \mathrm{d}p(V_g)$. In coordinates, we have

$$X_g = \boldsymbol{w} g \partial_{\boldsymbol{w}}. \tag{5.6}$$

We note that generally, X_g is nonlinear with respect to w^i and f^j. The commutation relations between vector fields of such type are inherited from the relations in Lie algebra $\mathfrak{g}$:

$$[\![X_g, X_f]\!] = [\![\mathrm{d}p(V_g), \mathrm{d}p(V_f)]\!] = \mathrm{d}p([\![V_g, V_f]\!]) = \mathrm{d}p(V_{[\![g,f]\!]}) = X_{[\![g,f]\!]}.$$

We now take X_g for the representation $\varrho(g)$ of elements g of Lie superalgebra $\mathfrak{g}$; see [109] for other examples of representations of Lie algebras by using vector fields.

For the sake of definition we now set $n = 2$, $x^1 = x$, $x^2 = t$ and we take $k_0 = 1$, $k_1 = 0$ so that $w^1 = w$.

Using the representation ϱ we construct the prolongations of total derivatives,

$$\tilde{D}_x = \bar{D}_x + w_x \frac{\partial}{\partial w}, \qquad \tilde{D}_t = \bar{D}_t + w_t \frac{\partial}{\partial w},$$

by inspecting the way in which they act on the nonlocal variable w along W:

$$w_x = \bar{D}_x \lrcorner\, \mathrm{d}w, \qquad w_t = \bar{D}_t \lrcorner\, \mathrm{d}w.$$

We thus obtain[3] a one-dimensional covering $\tau\colon \tilde{\mathcal{E}} = W \times \mathcal{E}^\infty \to \mathcal{E}^\infty$ with nonlocal variable w.

Using representation (5.6) for the matrices A and B that determine the zero-curvature representation $\alpha^{\text{KdV}} = A\,\mathrm{d}x + B\,\mathrm{d}t$ for the KdV equation, we obtain their realizations in terms of the vector fields:

$$W_A = \frac{1}{\mu}(-\lambda w^2 + \mu^2 + u_{12} w^2)\,\partial/\partial w,$$
$$W_B = \frac{1}{\mu}(-u_{12;xx} w^2 - 2u_{12;x}\mu w + 4\lambda^2 w^2 - 4\lambda\mu^2 - 2\lambda u_{12} w^2 - 2\mu^2 u_{12} - 2u_{12}^2 w^2)\,\partial/\partial w.$$

Therefore, the prolongations of the total derivatives act on the nonlocality w as follows:

$$w_x = -\frac{1}{\mu}(-\lambda w^2 + \mu^2 + u_{12} w^2), \tag{5.7a}$$
$$w_t = -\frac{1}{\mu}(-u_{12;xx} w^2 - 2u_{12;x}\mu w + 4\lambda^2 w^2 - 4\lambda\mu^2 - 2\lambda u_{12} w^2 - 2\mu^2 u_{12} - 2u_{12}^2 w^2). \tag{5.7b}$$

The parameter μ is removable by the transformation $w \to \mu w$, which rescales it to unit. Applying this transformation to (5.7), we reproduce the covering (5.4).

Example 15 (A covering which is based on Gardner's deformation)**.** Consider the Gardner deformation [101] of the KdV equation (1.1),

$$\mathfrak{m}_\varepsilon = \left\{u_{12} = \tilde{u}_{12} - \varepsilon\tilde{u}_{12;x} - \varepsilon^2\tilde{u}_{12}^2\right\} : \mathcal{E}_\varepsilon \to \mathcal{E}_0, \tag{2.1a}$$
$$\mathcal{E}_\varepsilon = \left\{\tilde{u}_{12;t} = -(\tilde{u}_{12;xx} + 3\tilde{u}_{12}^2 - 2\varepsilon^2\tilde{u}_{12}^3)_x\right\}, \tag{2.1b}$$

Expressing $\tilde{u}_{12;x}$ from (2.1a) and substituting it in (2.1b), we obtain the one-dimensional covering over the KdV equation,

$$\tilde{u}_x = \frac{1}{\varepsilon}(\tilde{u}_{12} - u_{12}) - \varepsilon\tilde{u}_{12}^2, \tag{5.8a}$$
$$\tilde{u}_t = \frac{1}{\varepsilon}(u_{12;xx} + 2u_{12}^2) + \frac{1}{\varepsilon^2}u_{12;x} + \frac{1}{\varepsilon^3}u_{12} + \left(-2u_{12;x} - \frac{2}{\varepsilon}u_{12} - \frac{1}{\varepsilon^3}\right)\tilde{u}_{12} + \left(2\varepsilon u_{12} + \frac{1}{\varepsilon}\right)\tilde{u}_{12}^2, \tag{5.8b}$$

[3] Each zero-curvature representation with coefficients belonging to a matrix Lie algebra determines a (linear) covering, whereas each covering with fibre W can be regarded as a zero-curvature representation the coefficients of which take values in the Lie algebra of vector fields on W.

Indeed, let $x^1, \ldots, x^n$ be the independent variables in a given PDE and $\bar{D}_{x^i}$ be the corresponding total derivative operators. Then zero-curvature representations and coverings are described by the same equation (4.5),

$$[\bar{D}_{x^i} + A_i, \bar{D}_{x^j} + A_j] = 0, \qquad i, j = 1, \ldots, n.$$

In the case of zero-curvature representations, the coefficients A_i and A_j are functions with values in a Lie algebra. In the case of coverings, the objects A_i and A_j are vertical vector fields on the covering manifold. This correspondence between zero-curvature representations and coverings very often allows one to transfer results on ZCRs to results on coverings and *vice versa*. Lemma 2 and Proposition 7 in section 5.4 illustrate this general principle; similar results were considered in [49].

We claim that covering (5.8) is equivalent to the covering that was obtained in [12, p. 277] for the KdV equation. To prove this, we first put $\tilde{u}_{12} = -\tilde{v}/\varepsilon$. We have

$$-\frac{\tilde{v}}{\varepsilon} = -\frac{1}{\varepsilon^2}\tilde{v} - \frac{1}{\varepsilon}u_{12} - \frac{1}{\varepsilon}\tilde{v}^2,$$

in other words

$$\tilde{v}_x = u_{12} + \left(\tilde{v} + \frac{1}{2\varepsilon}\right)^2 - \frac{1}{4\varepsilon^2}.$$

Next, we put $p = \tilde{v} + 1/(2\varepsilon)$, whence we obtain

$$p_x = u_{12} + p^2 - \frac{1}{4\varepsilon^2}, \tag{5.9a}$$
$$p_t = -u_{12;xx} - 2u^2 - \frac{1}{2\varepsilon^2}u_{12} + \frac{1}{4\varepsilon^4} - 2u_{12;x}p - (2u_{12} + \frac{1}{\varepsilon^2})p^2. \tag{5.9b}$$

Dividing (5.4) by w^2, we conclude that

$$w_x = -1 + (\lambda - u_{12})w^2,$$
$$\frac{w_x}{w^2} = -\frac{1}{w^2} - u_{12} + \lambda.$$

On the other hand, we put $p = 1/w$, whence $p_x = -w_x/w^2$, and set $\lambda = 1/(4\varepsilon^2)$. This brings (5.4) to the same notation as in formulas (5.9),

$$p_x = u_{12} + p^2 - \lambda,$$
$$p_t = -u_{12;xx} - 2u_{12}^2 - 2\lambda u_{12} + 4\lambda^2 - 2u_{12;x}p - (2u_{12} + 4\lambda)p^2.$$

The corresponding one-form of the zero-curvature representation for the KdV equation is equal to

$$\alpha_2^{\mathrm{KdV}} = \begin{pmatrix} 0 & \lambda - u_{12} \\ 1 & 0 \end{pmatrix} \mathrm{d}x + \begin{pmatrix} -u_{12;x} & -4\lambda^2 + 2\lambda u_{12} + 2u_{12}^2 + u_{12;xx} \\ -4\lambda - 2u_{12} & u_{12;x} \end{pmatrix} \mathrm{d}t. \tag{5.10}$$

Below we show that this zero curvature representation is also equivalent to α^{KdV} from Example 11.

5.2 Gauge transformations and coverings

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G (so that $G = SL_2(\mathbb{C})$ in the previous example). Let us recall that for any-zero curvature representation α of a given equation $\mathcal{E}$ there exists the zero-curvature representation α^S such that

$$\alpha^S = \bar{d}S \cdot S^{-1} + S \cdot \alpha \cdot S^{-1}, \quad S \in C^\infty(\mathcal{E}^\infty, G). \tag{4.6}$$

The zero-curvature representation α^S is called *gauge-equivalent* to α and S is the *gauge transformation*. Suppose $\alpha = A_i\,\mathrm{d}x^i$. The gauge transformation S acts on the components A_i of α as follows

$$A_i^S = \tfrac{\mathrm{d}}{\mathrm{d}x^i}(S)S^{-1} + SA_iS^{-1}. \tag{4.6$'$}$$

Example 16 (The relation between the coverings which stem from gauge equivalent zero curvature representations)**.** Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $G = SL_2(\mathbb{C})$. Suppose $S \in C^\infty(\mathcal{E}^\infty, SL_2(\mathbb{C}))$, so that

$$S = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}, \quad \det S = 1.$$

Let $\alpha = \sum_i A_i\mathrm{d}x^i$ be a zero-curvature representation for an equation $\mathcal{E}$. Using decomposition (5.2) for $A_i \in C^\infty(\mathcal{E}^\infty)\otimes\mathfrak{sl}_2(\mathbb{C})$, we inspect how the gauge transformation S acts on the components of α:

$$\begin{aligned} A_i^S &= \tfrac{\mathrm{d}}{\mathrm{d}x^i}(S)S^{-1} + S(a_e^{(i)}\otimes\rho(e) + a_h^{(i)}\otimes\rho(h)a_e^{(i)}\otimes\rho(f))S^{-1} = \\ &= \tfrac{\mathrm{d}}{\mathrm{d}x^i}(S)S^{-1} + a_e^{(i)}\otimes(S\cdot\rho(e)\cdot S^{-1}) + a_h^{(i)}\otimes(S\cdot\rho(h)\cdot S^{-1}) + a_e^{(i)}\otimes(S\cdot\rho(f)\cdot S^{-1}). \end{aligned}$$

We have that

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}x^i}(S)S^{-1} &= \begin{pmatrix} s_{1;i}s_4 - s_{2;i}s_3 & s_{2;i}s_1 - s_{1;i}s_2 \\ s_{3;i}s_4 - s_{4;i}s_3 & s_{4;i}s_1 - s_{3;i}s_2 \end{pmatrix} = \begin{pmatrix} s_{1;i}s_4 - s_{2;i}s_3 & s_{2;i}s_1 - s_{1;i}s_2 \\ s_{3;i}s_4 - s_{4;i}s_3 & -s_{1;i}s_4 + s_{2;i}s_3 \end{pmatrix} = \\ &= (s_{2;i}s_1 - s_{1;i}s_2)\rho(e) + (s_{1;i}s_4 - s_{2;i}s_3)\rho(h) + (s_{3;i}s_4 - s_{4;i}s_3)\rho(f), \\ S\cdot\rho(e)\cdot S^{-1} &= \begin{pmatrix} -s_1s_3 & s_1^2 \\ -s_3^2 & s_1s_3 \end{pmatrix} = (s_1^2)\rho(e) + (-s_1s_3)\rho(h) + (-s_3^2)\rho(f), \\ S\cdot\rho(h)\cdot S^{-1} &= \begin{pmatrix} s_1s_4 + s_2s_3 & -2s_1s_2 \\ 2s_3s_4 & -s_1s_4 - s_2s_3 \end{pmatrix} = (-2s_1s_2)\rho(e) + (s_1s_4 + s_2s_3)\rho(h) + (2s_3s_4)\rho(f), \\ S\cdot\rho(f)\cdot S^{-1} &= \begin{pmatrix} s_2s_4 & -s_2^2 \\ s_4^2 & -s_2s_4 \end{pmatrix} = (-s_2^2)\rho(e) + (s_2s_4)\rho(h) + (s_4^2)\rho(f). \end{aligned}$$

We finally obtain

$$\begin{aligned} A_i^S = (s_{2;i}s_1 - s_{1;i}s_2 + s_1^2a_e^{(i)} - 2s_1s_2a_h^{(i)} - s_2^2a_f^{(i)})\otimes\rho(e) + \\ + (s_{1;i}s_4 - s_{2;i}s_3 - s_1s_3a_e^{(i)} + (s_1s_4 + s_2s_3)a_h^{(i)} + s_2s_4a_f^{(i)})\otimes\rho(h) + \\ + (s_{3;i}s_4 - s_{4;i}s_3 - s_3^2a_e^{(i)} + 2s_3s_4a_h^{(i)} + s_4^2a_f^{(i)})\otimes\rho(f). \end{aligned}$$

Passing to the vector field representation of A_i^S by using formula (5.2$'$), we have

$$\begin{aligned} V_{A_i^S} = (s_{2;i}s_1 - s_{1;i}s_2 + s_1^2a_e^{(i)} - 2s_1s_2a_h^{(i)} - s_2^2a_f^{(i)})\otimes\varrho(e) + \\ + (s_{1;i}s_4 - s_{2;i}s_3 - s_1s_3a_e^{(i)} + (s_1s_4 + s_2s_3)a_h^{(i)} + s_2s_4a_f^{(i)})\otimes\varrho(h) + \\ + (s_{3;i}s_4 - s_{4;i}s_3 - s_3^2a_e^{(i)} + 2s_3s_4a_h^{(i)} + s_4^2a_f^{(i)})\otimes\varrho(f). \end{aligned} \tag{5.11}$$

In other words, whenever we start from the covering of $\mathcal{E}$ associated with a zero-curvature representation α, such that the differentiation rules for the nonlocality w are

$$\tfrac{\mathrm{d}}{\mathrm{d}x^i}(w) = -a_e^{(i)} + 2a_h^{(i)} w + a_f^{(i)} w^2,$$

we obtain the covering which is associated with α^S:

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}x^i}(w_S) = -(s_{2;i}s_1 - s_{1;i}s_2 + s_1^2 a_e^{(i)} - 2s_1 s_2 a_h^{(i)} - s_2^2 a_f^{(i)}) +& \\ + 2(s_{1;i}s_4 - s_{2;i}s_3 - s_1 s_3 a_e^{(i)} + (s_1 s_4 + s_2 s_3) a_h^{(i)} + s_2 s_4 a_f^{(i)}) w_S +& \\ + (s_{3;i}s_4 - s_{4;i}s_3 - s_3^2 a_e^{(i)} + 2s_3 s_4 a_h^{(i)} + s_4^2 a_f^{(i)}) w_S^2.&\end{aligned} \tag{5.12}$$

We shall use this relation between the two coverings in the search of gauge transformations between known zero-curvature representations for the KdV equation.

Example 17 (Gauge transformations between zero-curvature representations for the KdV equation)**.** Let us find the gauge transformations that bring coverings (5.4) and (5.8) to the form (5.9).

For the transformation (5.4)→(5.9) we have

$$\begin{aligned}p_x = u_{12} + p^2 - \lambda = -(s_{2;x}s_1 - s_{1;x}s_2 + s_1^2 - s_2^2(\lambda - u_{12}) +& \\ - 2(s_{1;x}s_4 - s_{2;x}s_3 - s_1 s_3 + s_2 s_4(\lambda - u_{12}))p -& \\ - (s_{3;x}s_4 - s_{4;x}s_3 - s_3^2 + s_4^2(\lambda - u_{12}))p^2).&\end{aligned}$$

Solving this equation for s_i, we find a unique solution $s_2 = s_3 = \boldsymbol{i}$, $s_1 = s_4 = 0$:

$$S = \begin{pmatrix} 0 & \boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 0 & -\boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{pmatrix}. \tag{5.13}$$

The matrices of the zero curvature representations corresponding to the coverings (5.4) and (5.9) are related as follows:

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda - u_{12} & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda - u_{12} \\ 1 & 0 \end{pmatrix}.$$

On the other hand, for the transformation (5.8)→(5.9) we have

$$\begin{aligned}p_x = u_{12} + p^2 - \frac{1}{4\varepsilon^2} = -(s_{2;x}s_1 - s_{1;x}s_2 - s_1^2 \frac{u_{12}}{\varepsilon} + s_1 s_2 \frac{1}{\varepsilon} - s_2^2 \varepsilon +& \\ - 2(s_{1;x}s_4 - s_{2;x}s_3 + s_1 s_3 \frac{u_{12}}{\varepsilon} - (s_1 s_4 + s_2 s_3)\frac{1}{2\varepsilon} + s_2 s_4 \varepsilon)p -& \\ - (s_{3;x}s_4 - s_{4;x}s_3 + s_3^2 \frac{u_{12}}{\varepsilon} - s_3 s_4 \frac{1}{\varepsilon} + s_4^2 \varepsilon)p^2).&\end{aligned}$$

Solving this equation for s_i, we find a solution $s_1 = \boldsymbol{i}/\sqrt{\varepsilon}$, $s_2 = \boldsymbol{i}/(2\varepsilon\sqrt{\varepsilon})$, $s_3 = 0$, $s_4 = \boldsymbol{i}\sqrt{\varepsilon}$. Therefore,

$$S = \begin{pmatrix} \boldsymbol{i}/\sqrt{\varepsilon} & \boldsymbol{i}/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & -\boldsymbol{i}\sqrt{\varepsilon} \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -\boldsymbol{i}\sqrt{\varepsilon} & -\boldsymbol{i}/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & \boldsymbol{i}/\sqrt{\varepsilon} \end{pmatrix}, \tag{5.14}$$

The matrices of the zero-curvature representations corresponding to coverings (5.8) and (5.9) satisfy the relation

$$\begin{pmatrix} \boldsymbol{i}/\sqrt{\varepsilon} & \boldsymbol{i}/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & -\boldsymbol{i}\sqrt{\varepsilon} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4\varepsilon^2} - u_{12} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\boldsymbol{i}\sqrt{\varepsilon} & -\boldsymbol{i}/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & \boldsymbol{i}/\sqrt{\varepsilon} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\varepsilon} & \frac{u_{12}}{\varepsilon} \\ -\varepsilon & -\frac{1}{2\varepsilon} \end{pmatrix}.$$

Let us remember that in Example 11 we derived the zero-curvature representation for the KdV equation from its Lax pair. Having done that, we also revised the transition from this zero-curvature representation to the Gardner deformation of the KdV equation. In what follows we extend this approach and find the generalizations of Gardner's deformation (2.1) for Krasil'shchik–Kersten system and for graded systems, in particular, for the N=1 and N=2 supersymmetric Korteweg–de Vries equations.

Example 18 (Gardner's deformation of Krasil'shchik–Kersten system)**.** Let us consider the Krasil'shchik–Kersten's system, which is the bosonic limit of (3.5) for a=1:

$$u_{12;t} = -u_{12;xxx} + 6u_{12}u_{12;x} - 3u_0u_{0;xxx} - 3u_{0;x}u_{0;xx} + 3u_{12;x}u_0^2 + 6u_{12}u_0u_{0;x}, \tag{5.15a}$$

$$u_{0;t} = -u_{0;xxx} + 3u_0^2u_{0;x} + 3u_{12}u_{0;x} + 3u_{12;x}u_0. \tag{5.15b}$$

Krasil'shchik–Kersten system (5.15) admits [53] an $\mathfrak{sl}_3(\mathbb{C})$-valued zero-curvature representation $\alpha_1^{\text{KK}} = A_1^{\text{KK}}\,\mathrm{d}x + B_1^{\text{KK}}\,\mathrm{d}t$ such that

$$A_1^{\text{KK}} = \begin{pmatrix} \eta & u_{12} - u_0^2 + 9\eta^2 & u_0 \\ 1 & \eta & 0 \\ 0 & 6\eta u_0 & -2\eta \end{pmatrix},$$

$$B_1^{\text{KK}} = \begin{pmatrix} b_{11} & b_{12} & -18\eta^2u_0 - 3\eta u_{0;x} - u_{0;xx} + u_0^3 + 2u_0u_{12} \\ -36\eta^2 + u_0^2 + 2u_{12} & -b_{11} - 72\eta^3 - 6\eta u_0^2 & -6\eta u_0 - u_{0;x} \\ -36\eta^2u_0 + 6\eta u_{0;x} & b_{32} & 72\eta^3 - 6\eta u_0^2 \end{pmatrix},$$

where the elements b_{11}, b_{12}, and b_{32} of the matrix B_1^{KK} are as follows:

$$\begin{aligned} b_{11} &= -36\eta^3 + 3\eta u_0^2 + u_{0;x}u_0 + u_{12;x}, \\ b_{12} &= -324\eta^4 + 9\eta^2(u_0^2 - 2u_{12}) - u_{0;xx}u_0 - u_{0;x}^2 - u_{12;xx} - u_0^4 - u_0^2u_{12} + 2u_{12}^2, \\ b_{32} &= -108\eta^3u_0 + 18\eta^2u_{0;x} + 6\eta(-u_{0;xx} + u_0^3 + 2u_0u_{12}). \end{aligned}$$

Let us construct the matrix $S^{\text{KK}} \in SL_3(\mathbb{C}) \hookrightarrow C^\infty(\mathcal{E}^\infty, SL_3(\mathbb{C}))$ for gauge transformation (5.14). We set $\varepsilon = \eta^2$ in expression for (5.14) and obtain

$$S^{\text{KK}} = \begin{pmatrix} \boldsymbol{i}\eta^{-1} & \boldsymbol{i}\frac{1}{2}\eta^{-3} & 0 \\ 0 & -\boldsymbol{i}\eta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By applying the gauge transformation S^{KK} to the zero-curvature representation α_1^{KK}, we obtain the gauge-equivalent zero-curvature representation α_2^{KK} for Krasil'shchik–Kersten system (5.15):

$$\alpha_2^{\mathrm{KK}} = (\alpha_1^{\mathrm{KK}})^{S^{\mathrm{KK}}} = A_2^{\mathrm{KK}}\,\mathrm{d}x + B_2^{\mathrm{KK}}\,\mathrm{d}t,$$

such that

$$A_2^{\mathrm{KK}} = \begin{pmatrix} \frac{2}{3}\eta^{-2} & u_0^2 - u_{12} & \boldsymbol{i}\eta^{-1}u_0 \\ -1 & -\frac{1}{3}\eta^{-2} & 0 \\ 0 & \boldsymbol{i}\eta^{-1}u_0 & -\frac{1}{3}\eta^{-2} \end{pmatrix},$$

$$B_2^{\mathrm{KK}} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ -u_0^2 - 2u_{12} + \eta^{-4} & -u_{0;x}u_0 - u_{12;x} - \eta^{-2}u_{12} + \frac{1}{3}\eta^{-6} & \boldsymbol{i}\eta^{-1}u_{0;x} + \boldsymbol{i}\eta^{-3}u_0 \\ -\boldsymbol{i}\eta^{-1}u_{0;x} + \boldsymbol{i}\eta^{-3}u_0 & \eta^{-1}(-\boldsymbol{i}u_{0;xx} + \boldsymbol{i}u_0^3 + 2\boldsymbol{i}u_0u_{12}) & -\eta^{-2}u_0^2 + \frac{1}{3}\eta^{-6} \end{pmatrix},$$

where the elements b_{11}, b_{12}, and b_{13} of the matrix B_2^{KK} are these:

$$\begin{aligned} b_{11} &= u_{0;x}u_0 + u_{12;x} + \eta^{-2}(u_0^2 + u_{12}) - \tfrac{2}{3}\eta^{-6}, \\ b_{12} &= u_{0;xx}u_0 + u_{0;x}^2 + u_{12;xx} + u_0^4 + u_0^2u_{12} - 2u_{12}^2 + \eta^{-2}(u_{0;x}u_0 + u_{12;x}) + \eta^{-4}u_{12}, \\ b_{13} &= \eta^{-1}(-\boldsymbol{i}u_{0;xx} + \boldsymbol{i}u_0^3 + 2\boldsymbol{i}u_0u_{12}) - \boldsymbol{i}\eta^{-3}u_{0;x} - \boldsymbol{i}\eta^{-5}u_0. \end{aligned}$$

Let us recall that formula (5.6) yields the representation of matrices A_2^{KK} and B_2^{KK} in terms of vector fields. By this argument, from the zero-curvature representation α_2^{KK} we obtain the two-dimensional covering over Krasil'shchik–Kersten system (5.15); denoting the new nonlocal variables by $\tilde{u}_0$ and $\tilde{u}_{12}$, we have that their derivatives are equal to

$$\tilde{u}_{0;x} = -\tilde{u}_0\tilde{u}_{12} - \boldsymbol{i}\eta^{-1}u_0 + \eta^{-2}\tilde{u}_0, \tag{5.16a}$$

$$\tilde{u}_{12;x} = -\tilde{u}_{12}^2 - u_0^2 + u_{12} - \boldsymbol{i}\eta^{-1}\tilde{u}_0u_0 + \eta^{-2}\tilde{u}_{12}, \tag{5.16b}$$

and

$$\begin{aligned} \tilde{u}_{0;t} = {}& \tilde{u}_0(u_{0;x}u_0 + u_{12;x} - \tilde{u}_{12}u_0^2 - 2\tilde{u}_{12}u_{12}) + \eta^{-1}(\boldsymbol{i}u_{0;xx} - \boldsymbol{i}u_{0;x}\tilde{u}_0^2 - \boldsymbol{i}u_{0;x}\tilde{u}_{12} - \boldsymbol{i}u_0^3 - 2\boldsymbol{i}u_0u_{12}) \\ & + \eta^{-2}\tilde{u}_0(2u_0^2 + u_{12}) + \eta^{-3}(\boldsymbol{i}u_{0;x} + \boldsymbol{i}\tilde{u}_0^2u_0 - \boldsymbol{i}\tilde{u}_{12}u_0) + \eta^{-4}\tilde{u}_0\tilde{u}_{12} + \boldsymbol{i}\eta^{-5}u_0 - \eta^{-6}\tilde{u}_0, \end{aligned} \tag{5.16c}$$

$$\begin{aligned} \tilde{u}_{12;t} = {}& -u_{0;xx}u_0 - u_{0;x}^2 + 2u_{0;x}\tilde{u}_{12}u_0 - u_{12;xx} + 2u_{12;x}\tilde{u}_{12} - \tilde{u}_{12}^2u_0^2 - 2\tilde{u}_{12}^2u_{12} - u_0^4 - u_0^2u_{12} \\ & + 2u_{12}^2 + \eta^{-1}\tilde{u}_0(\boldsymbol{i}u_{0;xx} - \boldsymbol{i}u_{0;x}\tilde{u}_{12} - \boldsymbol{i}u_0^3 - 2\boldsymbol{i}u_0u_{12}) \\ & + \eta^{-2}(-u_{0;x}u_0 - u_{12;x} + \tilde{u}_{12}u_0^2 + 2\tilde{u}_{12}u_{12}) \\ & + \boldsymbol{i}\eta^{-3}\tilde{u}_0\tilde{u}_{12}u_0 + \eta^{-4}(\tilde{u}_{12}^2 - u_{12}) - \eta^{-6}\tilde{u}_{12} \end{aligned} \tag{5.16d}$$

We note that under reduction $u_0 = 0$ this covering retracts to Gardner's deformation (5.8) for KdV equation (1.1).

Theorem 4. There is a "semi-classical" Gardner's deformation[4] for Krasil'shchik–Kersten equation (5.15). Under reduction $u_0 = 0$, this deformation contains classical Gardner's formulas (2.1). The Miura contraction from (5.18) to (5.15) is

$$u_0 = \tilde{u}_0 - \varepsilon\tilde{u}_{0;x} + \varepsilon^2\tilde{u}_{12}\tilde{u}_0, \tag{5.17a}$$

$$u_{12} = \tilde{u}_{12} - \varepsilon(\tilde{u}_{12;x} + \tilde{u}_{0;x}\tilde{u}_0) + \varepsilon^2(\tilde{u}_{0;x}^2 + \tilde{u}_{12}^2 + \tilde{u}_{12}\tilde{u}_0^2) - 2\varepsilon^3 u_{0;x}\tilde{u}_{12}\tilde{u}_0 + \varepsilon^4\tilde{u}_{12}^2\tilde{u}_0^2 \tag{5.17b}$$

The extension $\mathcal{E}(\varepsilon)$ of (5.15) is

$$\begin{aligned}\tilde{u}_{0;t} &= 3\varepsilon^4\tilde{u}_0^2\tilde{u}_{12}(2\tilde{u}_{0;x}\tilde{u}_{12} + \tilde{u}_{12;x}\tilde{u}_0) + 3\varepsilon^3\tilde{u}_0(-\tilde{u}_{0;xx}\tilde{u}_0\tilde{u}_{12} - 3\tilde{u}_{0;x}^2\tilde{u}_{12} - \tilde{u}_{0;x}\tilde{u}_{12;x}\tilde{u}_0)\\ &\quad + 3\varepsilon^2(\tilde{u}_{0;xx}\tilde{u}_{0;x}\tilde{u}_0 + \tilde{u}_{0;x}^3 + 3\tilde{u}_{0;x}\tilde{u}_0^2\tilde{u}_{12} + \tilde{u}_{0;x}\tilde{u}_{12}^2 + \tilde{u}_{12;x}\tilde{u}_0^3 + \tilde{u}_{12;x}\tilde{u}_0\tilde{u}_{12})\\ &\quad + 3\varepsilon\tilde{u}_0(-\tilde{u}_{0;xx}\tilde{u}_0 - 2\tilde{u}_{0;x}^2) - \tilde{u}_{0;xxx} + 3\tilde{u}_{0;x}\tilde{u}_0^2 + 3\tilde{u}_{0;x}\tilde{u}_{12} + 3\tilde{u}_{12;x}\tilde{u}_0,\end{aligned} \tag{5.18a}$$

$$\begin{aligned}\tilde{u}_{12;t} &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(-3\varepsilon^4\tilde{u}_0^2\tilde{u}_{12}^3 + 3\varepsilon^3\tilde{u}_0\tilde{u}_{12}(\tilde{u}_{0;x}\tilde{u}_{12} - \tilde{u}_{12;x}\tilde{u}_0) + \varepsilon^2(3\tilde{u}_{0;xx}\tilde{u}_0\tilde{u}_{12} + 3\tilde{u}_{0;x}\tilde{u}_{12;x}\tilde{u}_0 - 2\tilde{u}_{12}^3\\ &\quad - 6\tilde{u}_0^2\tilde{u}_{12}^2) + 3\varepsilon(-\tilde{u}_{0;xx}\tilde{u}_{0;x} + \tilde{u}_{0;x}\tilde{u}_0\tilde{u}_{12} - \tilde{u}_{12;x}\tilde{u}_0^2) + 3\tilde{u}_{0;xx}\tilde{u}_0 + \tilde{u}_{12;xx} - 3\tilde{u}_0^2\tilde{u}_{12} - 3\tilde{u}_{12}^2\Big).\end{aligned} \tag{5.18b}$$

Proof. Let us express u_0 and u_{12} from (5.16a)-(5.16b) and substitute them in (5.16c)-(5.16d). We get

$$\begin{aligned}u_0 &= \boldsymbol{i}\eta(\tilde{u}_{0;x} + \tilde{u}_0\tilde{u}_{12}) - \boldsymbol{i}\eta^{-1}\tilde{u}_0,\\ u_{12} &= \eta^2(-\tilde{u}_{0;x}^2 - 2\tilde{u}_{0;x}\tilde{u}_0\tilde{u}_{12} - \tilde{u}_0^2\tilde{u}_{12}^2) + \tilde{u}_{0;x}\tilde{u}_0 + \tilde{u}_{12;x} + \tilde{u}_0^2\tilde{u}_{12} + \tilde{u}_{12}^2 - \eta^{-2}\tilde{u}_{12},\\ u_{0;t} &= 3\eta^2(-\tilde{u}_{0;xx}\tilde{u}_{0;x}\tilde{u}_0 - \tilde{u}_{0;xx}\tilde{u}_0^2\tilde{u}_{12} - \tilde{u}_{0;x}^3 - 3\tilde{u}_{0;x}^2\tilde{u}_0\tilde{u}_{12} - \tilde{u}_{0;x}\tilde{u}_{12;x}\tilde{u}_0^2 - 2\tilde{u}_{0;x}\tilde{u}_0^2\tilde{u}_{12}^2\\ &\quad - \tilde{u}_{12;x}\tilde{u}_0^3\tilde{u}_{12}) - \tilde{u}_{0;xxx} + 3\tilde{u}_{0;xx}\tilde{u}_0^2 + 6\tilde{u}_{0;x}^2\tilde{u}_0 + 9\tilde{u}_{0;x}\tilde{u}_0^2\tilde{u}_{12} + 3\tilde{u}_{0;x}\tilde{u}_{12}^2 + 3\tilde{u}_{12;x}\tilde{u}_0^3\\ &\quad + 3\tilde{u}_{12;x}\tilde{u}_0\tilde{u}_{12} - 3\eta^{-2}(\tilde{u}_{0;x}\tilde{u}_0^2 + \tilde{u}_{0;x}\tilde{u}_{12} + \tilde{u}_{12;x}\tilde{u}_0),\\ u_{12;t} &= 3\eta^2(\tilde{u}_{0;xxx}\tilde{u}_{0;x} + \tilde{u}_{0;xxx}\tilde{u}_0\tilde{u}_{12} + \tilde{u}_{0;xx}^2 + \tilde{u}_{0;xx}\tilde{u}_{0;x}\tilde{u}_{12} + 2\tilde{u}_{0;xx}\tilde{u}_{12;x}\tilde{u}_0 - \tilde{u}_{0;xx}\tilde{u}_0\tilde{u}_{12}^2\\ &\quad + \tilde{u}_{0;x}^2\tilde{u}_{12;x} - \tilde{u}_{0;x}^2\tilde{u}_{12}^2 + \tilde{u}_{0;x}\tilde{u}_{12;xx}\tilde{u}_0 - 2\tilde{u}_{0;x}\tilde{u}_0\tilde{u}_{12}^3 + \tilde{u}_{12;xx}\tilde{u}_0^2\tilde{u}_{12} + \tilde{u}_{12;x}^2\tilde{u}_0^2 - 3\tilde{u}_{12;x}\tilde{u}_0^2\tilde{u}_{12}^2)\\ &\quad - 3\tilde{u}_{0;xxx}\tilde{u}_0 - 3\tilde{u}_{0;xx}\tilde{u}_{0;x} + 3\tilde{u}_{0;xx}\tilde{u}_0\tilde{u}_{12} + 3\tilde{u}_{0;x}^2\tilde{u}_{12} - 3\tilde{u}_{0;x}\tilde{u}_{12;x}\tilde{u}_0 + 12\tilde{u}_{0;x}\tilde{u}_0\tilde{u}_{12}^2\\ &\quad - \tilde{u}_{12;xxx} - 3\tilde{u}_{12;xx}\tilde{u}_0^2 + 12\tilde{u}_{12;x}\tilde{u}_0^2\tilde{u}_{12} + 6\tilde{u}_{12;x}\tilde{u}_{12}^2\\ &\quad + 3\eta^{-2}(-2\tilde{u}_{0;x}\tilde{u}_0\tilde{u}_{12} - \tilde{u}_{12;x}\tilde{u}_0^2 - 2\tilde{u}_{12;x}\tilde{u}_{12})\end{aligned}$$

Setting now $\tilde{u}_{0;\mathrm{new}} = \boldsymbol{i}\eta\tilde{u}_{0;\mathrm{old}}$ and $\tilde{u}_{12;\mathrm{new}} = \eta^2\tilde{u}_{12;\mathrm{old}}$ and denoting $\eta = \sqrt{\varepsilon}$, we obtain (5.17)-(5.18). □

Theorem 5. Gardner's deformation (5.17)-(5.18) for Krasil'shchik–Kersten system (5.15) yields recurrence relations the between conserved densities w_n; the relations are defined by

[4]See Remark 10 on page 74.

the formulas

$$
\begin{aligned}
w_0 &= u_{12}, \qquad w_1 = u_{12;x} + u_{0;x} u_0,\\
w_2 &= D_x w_1 + 2D_x(v_0 v_1) + u_{0;x}^2 + u_{12}^2 + u_{12} u_0^2,\\
w_3 &= D_x w_2 + \sum_{k=0}^{2} D_x(v_k v_{2-k}) + 2D_x(v_0 v_1) + 2w_1 w_0 + w_1 v_0^2 + 2w_0 v_1 v_0 + u_{12}^2 u_0^2,\\
w_n &= D_x w_{n-1} + \sum_{k=0}^{n-1} D_x(v_k v_{n-1-k}) + \sum_{k=0}^{n-2} (D_x(v_k) D_x(v_{n-2-k}) + w_k w_{n-2-k})\\
&\quad + \sum_{k+l+j=n-2} w_k v_l v_j + \sum_{k+l+j=n-3} w_k v_l D_x v_j + \sum_{k+l+j_i} w_k w_l v_j v_i,
\end{aligned}
$$

where v_i are given by

$$
v_0 = u_0, \qquad v_1 = u_{0;x}, \qquad v_n = D_x v_{n-1} + \sum_{k=0}^{n-2} w_k v_{n-2-k}.
$$

The generating function $\breve{w}(u_0, u_{12}, \varepsilon)$ of the zero differential order component of the series $w([u_0, u_{12}], \varepsilon)$ is given by the formula

$$
\breve{w} = \frac{12\varepsilon^2(-u_0^2 + u_{12}) + q^2 - 4q + 4}{6\varepsilon^2 q}, \tag{5.19}
$$

where

$$
\begin{aligned}
q = 2^{2/3}\Big(&9\varepsilon^2(2u_0^2 + u_{12}) + 2\\
&+ 3\sqrt{3}\varepsilon\sqrt{4\varepsilon^4(u_0^6 - 3u_0^4 u_{12} + 3u_0^2 u_{12}^2 - u_{12}^3) + \varepsilon^2(8u_0^4 + 20u_0^2 u_{12} - u_{12}^2) + 4u_0^2}\Big)^{1/3}.
\end{aligned}
$$

Proof. Plugging the series $\tilde{u}_0 = \sum_{k=0}^{+\infty} \varepsilon^k v_k$ and $\tilde{u}_{12} = \sum_{k=0}^{+\infty} \varepsilon^k w_k$ into (5.18), we obtain the recurrence relations between v_k and w_k. The series coefficients w_k are conserved because $\tilde{u}_{12;t}$ is in divergent form (i.e., the image of $\mathrm{d}/\mathrm{d}x$). The series coefficients v_k are auxiliary quantities which are not conserved in the general case.

Consider the zero order components of (5.17). The following system of equations hold for $\breve{v}(u_0, u_{12}, \varepsilon)$ and $\breve{w}(u_0, u_{12}, \varepsilon)$:

$$
u_0 = \breve{v} + \varepsilon^2 \breve{w}, \tag{5.20}
$$

$$
u_{12} = \breve{w} + \varepsilon^2(\breve{v}^2 \breve{w} + \breve{w}^2) + \varepsilon^4 \breve{w}^2 \breve{v}^2. \tag{5.21}
$$

From (5.20) we express $\breve{v}$ and obtain

$$
\breve{v} = \frac{u_0}{1 + \varepsilon^2 \breve{w}},
$$

Substituting of this expression for $\breve{v}$ in (5.21), we obtain a third order algebraic equation in $\breve{w}$,

$$\varepsilon^4\breve{w}^3 + 2\varepsilon^2\breve{w}^2 + (\varepsilon^2u_0^2 - \varepsilon^2u_{12} + 1)\breve{w} - u_{12} = 0.$$

In agreement with the limit behaviour of its solution $\lim_{\varepsilon\to 0}\breve{w} = u_{12}$, we take root (5.19) of this equation. □

5.3 Zero-curvature representations of graded extensions of the KdV equation

The graded extension of Maurer–Cartan's equation (4.5) has the form

$$\tfrac{\mathrm{d}}{\mathrm{d}x^j}A_i - \tfrac{\mathrm{d}}{\mathrm{d}x^i}A_j + [A_i, A_j] = 0, \quad \forall i,j = 1,\ldots,m : i \neq j. \tag{5.22}$$

Let us study in more detail the geometry of N=1 and N=2 supersymmetry-invariant generalizations of the Korteweg–de Vries equation [84, 98].

$N = 1$ supersymmetric Korteweg–de Vries equation

The $N = 1$ supersymmetric generalization of the KdV equation (1.1) is the sKdV equation [98]

$$\phi_t = -\phi_{xxx} - 3(\phi\mathcal{D}\phi)_x, \quad \mathcal{D} = \frac{\partial}{\partial\theta} + \theta\frac{d}{\mathrm{d}x}, \tag{5.23}$$

where $\phi(x,t,\theta) = \xi + \theta u$ is a complex fermionic super-field, θ is the Grassmann (or anti-commuting) variable such that $\theta^2 = 0$, the unknown u is the bosonic field, and ξ is the fermionic field. By using the expansion $\phi(x,t,\theta) = \xi + \theta u$ in (5.23), we obtain

$$\underline{u_t} = \underline{-u_{xxx} - 6uu_x} + 3\xi\xi_{xx}, \tag{5.24a}$$

$$\xi_t = -\,\xi_{xxx} - 3(u\xi)_x. \tag{5.24b}$$

The KdV equation (1.1) is underlined in (5.24a).

Example 19 (Zero-curvature representation and Gardner's deformation of the sKdV equation)**.** The N=1 sKdV equation (5.24) admits the $\mathfrak{sl}(2 \mid 1)$-valued zero-curvature representation

$$\alpha^{N=1} = A_1^{N=1}\mathrm{d}x + B_1^{N=1}\mathrm{d}t,$$

where

$$A_1^{N=1} = \begin{pmatrix} -\frac{1}{2\varepsilon} & -u + \frac{1}{4\varepsilon^2} & \xi \\ 1 & -\frac{1}{2\varepsilon} & 0 \\ 0 & -\xi & -\frac{1}{\varepsilon} \end{pmatrix},$$

$$B_1^{N=1} = \begin{pmatrix} \frac{1}{2}\varepsilon^{-3} - u_x & 2u^2 + u_{xx} - \xi\xi_x + \frac{1}{2}\varepsilon^{-2}u - \frac{1}{4}\varepsilon^{-4} & -\xi_{xx} - 2\xi u - \frac{1}{2}\varepsilon^{-1}\xi_x - \frac{1}{2}\varepsilon^{-2}\xi \\ -2u - \varepsilon^{-2} & \frac{1}{2}\varepsilon^{-3} + u_x & -\xi_x - \xi\varepsilon^{-1} \\ -\xi_x + \xi\varepsilon^{-1} & \xi_{xx} + 2\xi u - \frac{1}{2}\varepsilon^{-1}\xi_x + \frac{1}{2}\varepsilon^{-2}\xi & \varepsilon^{-3} \end{pmatrix}.$$

This zero-curvature can be obtained by reduction in zero-curvature representation (5.27) which we will consider below for N=2, a = 4-SKdV (1.2). Simultaneously, this zero-curvature representation is a generalisation of zero-curvature representation (5.10) for Korteweg-de Vires equation (1.1).

Let us construct the generalisation $S^{N=1} \in SL(2 \mid 1) \hookrightarrow C^\infty(\mathcal{E}^\infty, SL(2 \mid 1))$ of gauge transformation (5.14) where we had $S \in SL_2(\mathbb{C}) \simeq SL(2 \mid 0)$. Taking into account Remark 5, we consider the ansatz $S^{N=1} = S_+^{N=1} S_0^{N=1} S_-^{N=1}$, where $S_\nu \in G_\nu$, $\nu \in \{+, 0, -\}$. Bearing in mind that $SL_2(\mathbb{C}) \simeq SL(2 \mid 0) \subset GL(2 \mid 0)$, we construct S by using the following scheme:

1. we obtain an element $S_0^{N=1}$ by the multiplication of S from right and left by some matrices from $GL(2|0)$;
2. we specify the matrices $S_+^{N=1}$ and $S_-^{N=1}$.

We construct the matrix $S^{N=1}$ as follows

$$S^{N=1} = \begin{pmatrix} -1 & -\frac{1}{2}\varepsilon^{-1} & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S_+^{N=1}} \underbrace{\begin{pmatrix} \boldsymbol{i}\sqrt{\varepsilon} & \boldsymbol{i}\sqrt{\varepsilon}/\varepsilon^2 & 0 \\ 0 & \boldsymbol{i}\sqrt{\varepsilon} & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} \overbrace{\begin{pmatrix} \boldsymbol{i}/\sqrt{\varepsilon} & \boldsymbol{i}/(2\varepsilon\sqrt{\varepsilon}) & 0 \\ 0 & -\boldsymbol{i}\sqrt{\varepsilon} & 0 \\ 1 & 0 & 1 \end{pmatrix}}^{S} \begin{pmatrix} 1 & \varepsilon^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S_0^{N=1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S_-^{N=1}}. \tag{5.25}$$

By applying the gauge transformation $S^{N=1}$ to the zero-curvature representation $\alpha^{N=1}$, we obtain the gauge-equivalent zero-curvature representation β for sKdV equation (5.24):

$$\beta^{N=1} = (\alpha^{N=1})^{S^{N=1}} = A_2^{N=1}\mathrm{d}x + B_2^{N=1}\mathrm{d}t, \tag{5.26}$$

where

$$A_2^{N=1} = \begin{pmatrix} 0 & \varepsilon^{-1}u & \varepsilon^{-1}\xi \\ -\varepsilon & -\varepsilon^{-1} & 0 \\ 0 & \xi & -\varepsilon^{-1} \end{pmatrix},$$

$$B_2^{N=1} = \begin{pmatrix} u_x - u\varepsilon^{-1} & \frac{1}{\varepsilon}(-2u^2 - u_{xx} + \xi\xi_x) - \frac{1}{\varepsilon^2}u_x - \frac{1}{\varepsilon^3}u & \frac{1}{\varepsilon}(-\xi_{xx} - 2\xi u) - \frac{1}{\varepsilon^2}\xi_x - \frac{1}{\varepsilon^3}\xi \\ 2u\varepsilon + \varepsilon^{-1} & u_x + u\varepsilon^{-1} + \varepsilon^{-3} & \xi_x + \xi\varepsilon^{-1} \\ -\xi_{;x}\varepsilon + \xi & -\xi_{xx} - 2\xi u & \varepsilon^{-3} \end{pmatrix}.$$

Let us recall that formula (5.6) yields the representation of the matrices $A_2^{N=1}$ and $B_2^{N=1}$ in terms of vector fields. By this argument, from the zero-curvature representation $\beta^{N=1}$ we obtain the two-dimensional covering over sKdV equation (5.24); one of the two new nonlocal variables is bosonic (let us denote it by $\tilde{u}$) and the other, $\tilde{\xi}$ is fermionic:

$$
\begin{aligned}
\tilde{u}_x &= -\tilde{u}^2\varepsilon + (\tilde{u} - u)\varepsilon^{-1} - \tilde{\xi}\xi,\\
\tilde{\xi}_x &= -\tilde{\xi}\tilde{u}\varepsilon + (\tilde{\xi} - \xi)\varepsilon^{-1},\\
\tilde{u}_t &= \frac{1}{\varepsilon^3}(2\tilde{u}^2u\varepsilon^4 + \tilde{u}^2\varepsilon^2 - 2\tilde{u}u\varepsilon^2 - 2\tilde{u}u_x\varepsilon^3 - \tilde{u} + 2u^2\varepsilon^2 + u + u_{xx}\varepsilon^2 + u_x\varepsilon - \tilde{\xi}\tilde{u}\xi_x\varepsilon^4 +\\
&\quad + \tilde{\xi}\xi_{xx}\varepsilon^3 + \tilde{\xi}\xi\tilde{u}\varepsilon^3 + 2\tilde{\xi}\xi u\varepsilon^3 - \xi\xi_x\varepsilon^2),\\
\tilde{\xi}_t &= \frac{1}{\varepsilon^3}(-\tilde{u}\xi_x\varepsilon^3 + \xi_{xx}\varepsilon^2 + \xi_x\varepsilon + 2\tilde{\xi}\tilde{u}u\varepsilon^4 + \tilde{\xi}\tilde{u}\varepsilon^2 - \tilde{\xi}u\varepsilon^2 - \tilde{\xi}u_x\varepsilon^3 - \tilde{\xi} - \xi\tilde{u}\varepsilon^2 + 2\xi u\varepsilon^2 + \xi).
\end{aligned}
$$

We now express the local variables u and ξ from $\tilde{u}_x$ and $\tilde{\xi}_x$ and substitute them in $\tilde{u}_t$ and $\tilde{\xi}_t$. We thus obtain the Gardner deformation [99] of sKdV equation (5.24):

$$
\begin{aligned}
\mathcal{E}_\varepsilon &= \Big\{\underline{\tilde{u}_t} = \underline{6\tilde{u}^2\tilde{u}_x\varepsilon^2 - 6\tilde{u}\tilde{u}_x - \tilde{u}_{xxx}} - 3\tilde{\xi}\tilde{u}\tilde{\xi}_{xx}\varepsilon^2 + 3\tilde{\xi}\tilde{\xi}xx - 3\tilde{\xi}\tilde{\xi}_x\tilde{u}_x\varepsilon^2,\\
&\qquad \tilde{\xi}_t = 3\tilde{u}^2\tilde{\xi}_x\varepsilon^2 - 3\tilde{u}\tilde{\xi}_x - \tilde{\xi}_{xxx} + 3\tilde{\xi}\tilde{u}\tilde{u}_x\varepsilon^2 - 3\tilde{\xi}\tilde{u}_x\Big\},\\
\mathfrak{m}_\varepsilon &= \Big\{\underline{u} = \underline{\tilde{u} - \varepsilon\tilde{u}_x} + \varepsilon^2(\tilde{\xi}\tilde{\xi}_x\varepsilon^2 \underline{- \tilde{u}^2}), \quad \xi = \tilde{\xi} - \varepsilon\tilde{\xi}_x - \varepsilon^2\tilde{\xi}\tilde{u}\Big\} \colon \mathcal{E}_\varepsilon \to \mathcal{E}_{\text{sKdV}}.
\end{aligned}
$$

This deformation can also be obtained by using super-field formalism [99]. The original Gardner deformation (2.1) of the KdV equation (1.1) is underlined in the above formulas.

$N = 2$ supersymmetric Korteweg–de Vries equation

Let us consider the four-component generalization of the KdV equation (1.1), namely, the N=2 supersymmetric Korteweg–de Vries equation (SKdV) [84]:

$$
\boldsymbol{u}_t = -\boldsymbol{u}_{xxx} + 3\big(\boldsymbol{u}\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}\big)_x + \frac{a-1}{2}\big(\mathcal{D}_1\mathcal{D}_2\boldsymbol{u}^2\big)_x + 3a\boldsymbol{u}^2\boldsymbol{u}_x, \qquad \mathcal{D}_i = \frac{\partial}{\partial\theta_i} + \theta_i \cdot \frac{\mathrm{d}}{\mathrm{d}x}, \tag{1.2}
$$

where

$$
\boldsymbol{u}(x, t; \theta_1, \theta_2) = u_0(x, t) + \theta_1 \cdot u_1(x, t) + \theta_2 \cdot u_2(x, t) + \theta_1\theta_2 \cdot u_{12}(x, t) \tag{3.2}
$$

is the complex bosonic super-field, θ_1, θ_2 are Grassmann variables such that $\theta_1^2 = \theta_2^2 = \theta_1\theta_2 + \theta_2\theta_1 = 0$, u_0, u_{12} are bosonic fields, and u_1, u_2 are fermionic fields. Expansion (3.2) converts (1.2) to the four-component system (3.5).

The Gardner deformation problem for the $N = 2$ supersymmetric $a = 4$ KdV equation was formulated in [84]. In Chapter 3 it was shown that one can not construct such a deformation under the assumptions that, first, the deformation is polynomial in $\mathcal{E}$, second, it involves only the super-fields but not their components, and third, it contains known deformation (2.1) under the reduction $u_0 = 0$, $u_1 = u_2 = 0$. Therefore, we shall find a graded generalization of Gardner's deformation (2.1) for the system of four equations (3.5) treating it in components but not as a single equation (1.2) upon the super-field.

The SKdV equation (3.5) admits [28] the $\mathfrak{sl}(2 \mid 1)$-valued zero-curvature representation $\alpha^{N=2} = A\mathrm{d}x + B\mathrm{d}t$ such that

$$A = \begin{pmatrix} \eta - \boldsymbol{i}u_0 & \eta^2 - 2\boldsymbol{i}\eta u_0 - u_0^2 - u_{12} & -u_2 - \boldsymbol{i}u_1 \\ 1 & \eta - \boldsymbol{i}u_0 & 0 \\ 0 & u_2 - \boldsymbol{i}u_1 & 2\eta - 2iu_0 \end{pmatrix}, \tag{5.27a}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{5.27b}$$

where the elements of B are as follows:

$$\begin{aligned}
b_{11} = {} & -4\eta^3 - 2\boldsymbol{i}\eta u_{0;x} - 4\boldsymbol{i}u_0^3 + 6\boldsymbol{i}u_0u_{12} + 4u_0u_{0;x} + \boldsymbol{i}u_{0;xx} - u_{12;x} + 4\boldsymbol{i}u_2u_1, \\
b_{12} = {} & -4\eta^4 + 4\boldsymbol{i}\eta^3u_0 + 2\eta^2u_{12} - 4\boldsymbol{i}\eta u_0^3 + 8\boldsymbol{i}\eta u_0u_{12} + 2\boldsymbol{i}\eta u_{0;xx} - 4u_0^4 - 2u_0^2u_{12} - \\
& - 4u_0u_{0;xx} + 2u_{12}^2 - 4u_{0;x}^2 + u_{12;xx} - u_2u_{2;x} + 4\boldsymbol{i}u_2u_1\eta - 8u_2u_1u_0 - u_1u_{1;x}, \\
b_{13} = {} & -\eta u_{2;x} - \boldsymbol{i}\eta u_{1;x} - 5\boldsymbol{i}u_0u_{2;x} + 5u_0u_{1;x} + u_{2;xx} + \boldsymbol{i}u_{1;xx} + 2u_2\eta^2 + 2\boldsymbol{i}u_2\eta u_0 - \\
& - 8u_2u_0^2 + 2u_2u_{12} - 4u_2u_{0;x}\boldsymbol{i} + 2u_1\eta^2\boldsymbol{i} - 2u_1\eta u_0 - 8u_1u_0^2\boldsymbol{i} + 2u_1u_{12}\boldsymbol{i} + 4u_1u_{0;x}, \\
b_{21} = {} & -4\eta^2 - 4\boldsymbol{i}\eta u_0 + 4u_0^2 - 2u_{12}, \\
b_{22} = {} & -4\eta^3 + 2\boldsymbol{i}\eta u_{0;x} - 4\boldsymbol{i}u_0^3 + 6\boldsymbol{i}u_0u_{12} - 4u_0u_{0;x} + \boldsymbol{i}u_{0;xx} + u_{12;x} + 4\boldsymbol{i}u_2u_1, \\
b_{23} = {} & u_{2;x} + \boldsymbol{i}u_{1;x} - 2u_2\eta - 4\boldsymbol{i}u_2u_0 - 2\boldsymbol{i}u_1\eta + 4u_1u_0, \\
b_{31} = {} & u_{2;x} - \boldsymbol{i}u_{1;x} + 2u_2\eta + 4\boldsymbol{i}u_2u_0 - 2\boldsymbol{i}u_1\eta + 4u_1u_0, \\
b_{32} = {} & -\eta u_{2;x} + \boldsymbol{i}\eta u_{1;x} - 5\boldsymbol{i}u_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} + \boldsymbol{i}u_{1;xx} - 2u_2\eta^2 - 2\boldsymbol{i}u_2\eta u_0 + \\
& + 8u_2u_0^2 - 2u_2u_{12} - 4\boldsymbol{i}u_2u_{0;x} + 2\boldsymbol{i}u_1\eta^2 - 2u_1\eta u_0 - 8\boldsymbol{i}u_1u_0^2 + 2\boldsymbol{i}u_1u_{12} - 4u_1u_{0;x}, \\
b_{33} = {} & -8\eta^3 - 8\boldsymbol{i}u_0^3 + 12\boldsymbol{i}u_0u_{12} + 2\boldsymbol{i}u_{0;xx} + 8\boldsymbol{i}u_2u_1.
\end{aligned}$$

In Example 9 we prove that the parameter $\eta \in \mathbb{C}$ is non-removable from A and B under gauge transformation.

Remark 9. Let us recall that the vectors Z, H and $E^{\pm}$ that belong to $\mathfrak{sl}(2 \mid 1)$ generate a basis in $\mathfrak{gl}(2, \mathbb{C})$ (see the respective formulas on p. 42 in Example 8 on page 42). We notice that the vector Z commutes with any other vector from $\mathfrak{gl}(2, \mathbb{C})$.

The reduction $u_0 = u_1 = u_2 = 0$ converts zero-curvature representation (5.27) to the

$\mathfrak{gl}(2,\mathbb{C})$-valued zero-curvature representation of the KdV equation (1.1),

$$A_{\text{KdV}} = \begin{pmatrix} \eta & \eta^2 - u_{12} & 0 \\ 1 & \eta & 0 \\ 0 & 0 & 2\eta \end{pmatrix},$$

$$B_{\text{KdV}} = \begin{pmatrix} -4\eta^3 - u_{12;x} & -4\eta^4 + 2\eta^2 u_{12} + 2u_{12}^2 + u_{12;xx} & 0 \\ 2(-2\eta^2 - u_{12}) & -4\eta^3 + u_{12;x} & 0 \\ 0 & 0 & -8\eta^3 \end{pmatrix}.$$

Taking into account Remark 9, we obtain the $\mathfrak{sl}(2,\mathbb{C})$-valued zero-curvature representation (5.10) for the KdV equation (1.1) by omitting the summands $\eta \otimes Z\mathrm{d}x$ and $-4\eta^3 \otimes Z\mathrm{d}t$ in A_{KdV} and B_{KdV} and by denoting $\eta^2 = \lambda$.

Proposition 6 ([65])**.** The N=2 supersymmetric a=4 Korteweg–de Vries equation (3.5) admits the $(1 \mid 1)$-dimensional $\mathbb{Z}_2$-graded covering, which is given in formulas (5.30–5.31) and which is such that, under the reduction $u_0 = u_1 = u_2 = 0$ of (3.5) to the KdV equation (1.1) and the consistent trivialization $f := 0$ in (5.30a–5.31a), see also (5.32), covering (5.30–5.31) reduces to the known Gardner deformation of (1.1) in the form of (5.8).

Proof. Let us extend the gauge transformation (5.14), which was determined by the element S of the Lie group $SL(2,\mathbb{C})$. We let

$$S^{N=2} = \begin{pmatrix} -1 & -\frac{1}{2}\varepsilon^{-1} & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix}. \tag{5.28}$$

Acting by gauge transformation (5.28) on zero-curvature representation (5.27), we obtain the graded zero-curvature representation that contains the "small" zero-curvature representation which, in turn, originates from (5.8) and is gauge-equivalent to (5.10) for the KdV equation (1.1). Specifically, we have that

$$A = \begin{pmatrix} iu_0 & \varepsilon^{-1}(u_0^2 + u_{12}) - \boldsymbol{i}\varepsilon^{-2}u_0 & \varepsilon^{-1}(u_2 - iu_1) \\ -\varepsilon & \boldsymbol{i}u_0 - \varepsilon^{-1} & 0 \\ 0 & u_2 + \boldsymbol{i}u_1 & 2iu_0 - \varepsilon^{-1} \end{pmatrix}, \tag{5.29a}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{5.29b}$$

where the elements of the matrix B are as follows,

$$
\begin{aligned}
b_{11} &= 4\boldsymbol{i}u_0^3 - 6\boldsymbol{i}u_0u_{12} + 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} - u_{12;x} - 4\boldsymbol{i}u_2u_1 + \varepsilon^{-1}(2u_0^2 - u_{12} - \boldsymbol{i}u_{0;x}) - \boldsymbol{i}\varepsilon^{-2}u_0,\\
b_{12} &= \varepsilon^{-1}(4u_0^4 + 2u_0^2u_{12} + 4u_0u_{0;xx} - 2u_{12}^2 + 4u_{0;x}^2 - u_{12;xx} + u_2u_{2;x} + 8u_2u_1u_0 + u_1u_{1;x}) +\\
&\quad + \varepsilon^{-2}(2\boldsymbol{i}u_0^3 - 4\boldsymbol{i}u_0u_{12} + 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} - u_{12;x} - 2\boldsymbol{i}u_2u_1) + \varepsilon^{-3}(u_0^2 - u_{12} - \boldsymbol{i}u_{0;x}) -\\
&\quad - \boldsymbol{i}\varepsilon^{-4}u_0,\\
b_{13} &= \varepsilon^{-1}(-5\boldsymbol{i}u_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} + \boldsymbol{i}u_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} - 4\boldsymbol{i}u_2u_{0;x} - 8\boldsymbol{i}u_1u_0^2 +\\
&\quad + 2\boldsymbol{i}u_1u_{12} - 4u_1u_{0;x}) + \varepsilon^{-2}(-u_{2;x} + \boldsymbol{i}u_{1;x} - 3\boldsymbol{i}u_2u_0 - 3u_1u_0) + \varepsilon^{-3}(-u_2 + \boldsymbol{i}u_1),\\
b_{21} &= 2\varepsilon(-2u_0^2 + u_{12}) + 2\boldsymbol{i}u_0 + \varepsilon^{-1},\\
b_{22} &= 4\boldsymbol{i}u_0^3 - 6\boldsymbol{i}u_0u_{12} - 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} + u_{12;x} - 4\boldsymbol{i}u_2u_1 + \varepsilon^{-1}(-2u_0^2 + u_{12} + \boldsymbol{i}u_{0;x}) +\\
&\quad + \boldsymbol{i}\varepsilon^{-1}u_0 + \varepsilon^{-3},\\
b_{23} &= u_{2;x} - \boldsymbol{i}u_{1;x} + 4\boldsymbol{i}u_2u_0 + 4u_1u_0 + \varepsilon^{-1}(u_2 - \boldsymbol{i}u_1),\\
b_{31} &= \varepsilon(-u_{2;x} - \boldsymbol{i}u_{1;x} + 4\boldsymbol{i}u_2u_0 - 4u_1u_0) + u_2 + \boldsymbol{i}u_1,\\
b_{32} &= 5\boldsymbol{i}u_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} - \boldsymbol{i}u_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} + 4\boldsymbol{i}u_2u_{0;x} + 8\boldsymbol{i}u_1u_0^2 - 2\boldsymbol{i}u_1u_{12} -\\
&\quad - 4u_1u_{0;x} + \varepsilon^{-1}u_0(\boldsymbol{i}u_2 - u_1),\\
b_{33} &= 2(4\boldsymbol{i}u_0^3 - 6\boldsymbol{i}u_0u_{12} - \boldsymbol{i}u_{0;xx} - 4\boldsymbol{i}u_2u_1) + \varepsilon^{-3}.
\end{aligned}
$$

The projective substitution (5.5) yields the two-dimensional covering over the $N{=}2$, $a{=}4$ SKdV equation. Under the reduction $u_0 = u_1 = u_2 = 0$ the covering contains (5.8), which is equivalent to Gardner's deformation (2.1) of the KdV equation (1.1). The x-components of the derivation rules for the nonlocalites w and f are

$$
\begin{aligned}
\underline{w_x} &= \underline{-\varepsilon w^2 + \varepsilon^{-1}(w - u_{12})} - fu_2 - \boldsymbol{i}fu_1 - \varepsilon^{-1}u_0^2 - \varepsilon^{-2}\boldsymbol{i}u_0, && \text{(5.30a)}\\
f_x &= -\varepsilon wf - \boldsymbol{i}u_0f + \varepsilon^{-1}(f - u_2 + iu_1); && \text{(5.30b)}
\end{aligned}
$$

here and in what follows we underline covering (5.8) that encodes the "small" Gardner deformation for the KdV equation. The t-components of the "large" covering over the $N{=}2$, $a{=}4$ SKdV are

$$
\begin{aligned}
\underline{w_t} &= \varepsilon(-4w^2u_0^2 \underline{+ 2w^2u_{12}} - fwu_{2;x} - ifwu_{1;x} + 4\boldsymbol{i}fu_2wu_0 - 4fu_1wu_0) + 2\boldsymbol{i}w^2u_0 +\\
&\quad + 8wu_0u_{0;x} \underline{- 2wu_{12;x}} - 5\boldsymbol{i}fu_0u_{2;x} + 5fu_0u_{1;x} + fu_{2;xx} + \boldsymbol{i}fu_{1;xx} + fu_2w - 8fu_2u_0^2 +\\
&\quad + 2fu_2u_{12} - 4\boldsymbol{i}fu_2u_{0;x} + \boldsymbol{i}fu_1w - 8\boldsymbol{i}fu_1u_0^2 + 2\boldsymbol{i}fu_1u_{12} + 4fu_1u_{0;x} + \varepsilon^{-1}(\underline{w^2} + 4wu_0^2 -\\
&\quad \underline{- 2wu_{12}} - 2\boldsymbol{i}wu_{0;x} - 4u_0^4 - 2u_0^2u_{12} - 4u_0u_{0;xx} \underline{+ 2u_{12}^2} - 4u_{0;x}^2 \underline{+ u_{12;xx}} - \boldsymbol{i}fu_2u_0 +\\
&\quad + fu_1u_0 - u_2u_{2;x} - 8u_2u_1u_0 - u_1u_{1;x}) + \varepsilon^{-2}(-2\boldsymbol{i}wu_0 - 2\boldsymbol{i}u_0^3 + 4\boldsymbol{i}u_0u_{12} - 4u_0u_{0;x} +\\
&\quad + \boldsymbol{i}u_{0;xx} \underline{+ u_{12;x}} + 2\boldsymbol{i}u_2u_1) + \varepsilon^{-3}(\underline{- w} - u_0^2 \underline{+ u_{12}} + \boldsymbol{i}u_{0;x}) + \varepsilon^{-4}\boldsymbol{i}u_0, && \text{(5.31a)}
\end{aligned}
$$

$$
\begin{aligned}
f_t = {} & 2\varepsilon w(-2fu_0^2 + fu_{12}) + (-wu_{2;x} + \boldsymbol{i}wu_{1;x} + 2\boldsymbol{i}fwu_0 - 4\boldsymbol{i}fu_0^3 + 6\boldsymbol{i}fu_0u_{12} + 4fu_0u_{0;x} + \\
& + \boldsymbol{i}fu_{0;xx} - fu_{12;x} + 4\boldsymbol{i}fu_2u_1 - 4\boldsymbol{i}u_2wu_0 - 4u_1wu_0) + \varepsilon^{-1}(5\boldsymbol{i}u_0u_{2;x} + 5u_0u_{1;x} + u_{2;xx} - \\
& - \boldsymbol{i}u_{1;xx} + fw + 2fu_0^2 - fu_{12} - \boldsymbol{i}fu_{0;x} - u_2w - 8u_2u_0^2 + 2u_2u_{12} + 4\boldsymbol{i}u_2u_{0;x} + \boldsymbol{i}u_1w + \\
& + 8\boldsymbol{i}u_1u_0^2 - 2\boldsymbol{i}u_1u_{12} + 4u_1u_{0;x}) + \varepsilon^{-2}(u_{2;x} - \boldsymbol{i}u_{1;x} - \boldsymbol{i}fu_0 + 3\boldsymbol{i}u_2u_0 + 3u_1u_0) + \\
& + \varepsilon^{-3}(-f + u_2 - \boldsymbol{i}u_1).
\end{aligned} \tag{5.31b}
$$

It is noteworthy that the reduction $u_0 = u_1 = u_2 = 0$ in (3.5) eliminates the presence of the fermionic variables f in (5.30a) and (5.31a) so that there remains only (5.8) in the bosonic sector:

$$w_x = -\varepsilon w^2 + \varepsilon^{-1}(w - u_{12}), \tag{5.32a}$$

$$w_t = 2\varepsilon w^2 u_{12} - 2wu_{12;x} + \varepsilon^{-1}(w^2 - 2wu_{12} + 2u_{12}^2 + u_{12;xx}), \tag{5.32b}$$

$$f_x = -\varepsilon wf + \varepsilon^{-1}f, \tag{5.32c}$$

$$f_t = 2\varepsilon wfu_{12} - fu_{12;x} + \varepsilon^{-1}f(u_{12} - w) - \varepsilon^{-3}f. \tag{5.32d}$$

This proves our claim. □

In contrast with Gardner's deformation of the N=1 sKdV equation (see Example 19 on p. 64), covering (5.30–5.31), which we obtain for N=2 supersymmetric a=2 KdV equation, can not be expressed in terms of the super-field. The reduction $u_0 = 0$, $u_1 = 0$ (and the change of notation $u_2 \to \xi$, $u_{12} \to u$) maps this covering over the N=2, a=4 SKdV equation to the covering which was constructed in Example 19 for the N=1 supersymmetric Korteweg–de Vries equation (5.23).

Theorem 6. There is a "semi-classical" Gardner's deformation[5] for the N=2, $a = 4$-SKdV equation (e.g., in component form (3.5)). Under reduction $u_0 = u_1 = u_2$ and trivialisation $f := 0$, this deformation contains classical Gardner's formulas (2.1). The Miura contraction taking solutions of (5.34) to solutions of (3.5) is

$$u_{12} = \varepsilon^{-1}\boldsymbol{i}u_0 + w - u_0^2 + \varepsilon(-w_x - 2\boldsymbol{i}fu_1) + \varepsilon^2(-w^2 + ff_x), \tag{5.33a}$$

$$u_2 = -f - \boldsymbol{i}u_1 + \varepsilon(f_x - \boldsymbol{i}fu_0) + \varepsilon^2 wf. \tag{5.33b}$$

The extension $\mathcal{E}(\varepsilon)$ of (3.5) is

$$
\begin{aligned}
w_t = {} & \frac{\mathrm{d}}{\mathrm{d}x}\Big(-2\varepsilon^{-2}u_0^2 + 3\varepsilon^{-1}(-2u_0u_{0;x} - \boldsymbol{i}fu_1) - 3w^2 + 6u_0^2w - w_{xx} + 3\boldsymbol{i}u_1f_x \\
& + 3ff_x + 9u_1u_0f + 3\varepsilon\boldsymbol{i}(fwu_1 - u_0ff_x) + 3\varepsilon^2(w^2 - wff_x)\Big),
\end{aligned} \tag{5.34a}
$$

$$
\begin{aligned}
f_t = {} & -3\boldsymbol{i}\varepsilon^{-1}(u_0f_x + fu_{0;x}) \\
& - 3wf_x + 15u_0^2f_x + 6\boldsymbol{i}u_0f_{0;xx} - f_{0;xxx} + 6\boldsymbol{i}f_xu_{0;x} + 15fu_0u_{0;x} - 3fw_x \\
& + 3\varepsilon(\boldsymbol{i}fwu_{0;x} + 3\boldsymbol{i}fu_0w_x + 2\boldsymbol{i}fu_1f_x) + 3\varepsilon^2(w^2f_x + fww_x).
\end{aligned} \tag{5.34b}
$$

[5]See Remark 10 on page 74.

Proof. This Gardner's deformation is given by covering (5.30)-(5.31). Formulas (5.33) are obtained by expressing u_{12} and u_2 from (5.30). Formulas (5.34) are obtained by plugging (5.33) in (5.31). □

Theorem 7. • Gardner's deformation (5.33)-(5.34) for the N=2, a=4-SKdV equation yields recurrence relations between the conserved densities w_i; relations are defined by the formulas

$$\begin{aligned}
w_{-1} &= -\boldsymbol{i}u_0, \qquad w_0 = u_{12} - \boldsymbol{i}u_{0;x},\\
w_1 &= -2\boldsymbol{i}u_0w_0 + D_xw_0 + 2\boldsymbol{i}f_0u_1,\\
w_n &= -2\boldsymbol{i}u_0w_{n-1} + D_xw_{n-1} + 2\boldsymbol{i}f_{n-1}u_1 + \sum_{k=0}^{n-2}(w_kw_{n-2-k} - f_kD_xf_{n-2-k}),
\end{aligned}$$

where the auxiliary quantities f_i are given by

$$\begin{aligned}
f_0 &= -u_2 - \boldsymbol{i}u_1,\\
f_1 &= D_xf_0 - 2\boldsymbol{i}u_0f_0,\\
f_n &= D_xf_{n-1} - 2\boldsymbol{i}u_0f_{n-1} + \sum_{k=0}^{n-2} w_kf_{n-2-k}.
\end{aligned}$$

- The conserved densities w_i are non-trivial for all integer $k \geqslant -1$.
- The generating function $\breve{w}(u_0, u_1, u_2, u_{12}, \varepsilon)$ of the zero order component of the series $w([u_0, u_1, u_2, u_{12}], \varepsilon)$ with differential-polynomial coefficients is given by the formula

$$\breve{w} = \frac{1}{6q^{1/3}}\left(-16u_0^2 - 12u_{12} + 8\varepsilon^{-1}\boldsymbol{i}u_0(q^{1/3}+2) + \varepsilon^{-2}(q^{2/3} + 4q^{1/3} + 4)\right), \tag{5.35}$$

where

$$\begin{aligned}
q &= 8\varepsilon^3\boldsymbol{i}(8u_0^3 + 9u_0u_{12} - 27u_1u_2) - 48\varepsilon\boldsymbol{i}u_0 - 8\\
&+12\varepsilon^2\Big[8u_0^2{+}3u_{12}\Big(24\varepsilon^{-1}\boldsymbol{i}u_1u_23(-48u_0u_1u_2{-}u_{12}^2){+}12\varepsilon\boldsymbol{i}(-24u_0^2u_1u_2{-}u_0u_{12}^2{-}9u_1u_{12}u_2)\\
&\qquad + 12\varepsilon^2(16u_0^3u_1u_2 + u_0^2u_{12}^2 + 18u_0u_1u_{12}u_2 + u_{12}^3)\Big)^{\frac{1}{2}}\Big].
\end{aligned}$$

Proof. Let us substitute $w = \bar{w} - \boldsymbol{i}\varepsilon u_0$ into (5.33) (i.e., we put $w_{-1} = -\boldsymbol{i}u_0$ in expansion $w = \sum_{k=-1}^{+\infty}\varepsilon^kw_k$). We have

$$u_{12} = \bar{w} + \boldsymbol{i}u_{0;x} + \varepsilon\left(-\bar{w}_x - 2\boldsymbol{i}fu_1 + 2\boldsymbol{i}u_0\bar{w}\right) + \varepsilon^2\left(-\bar{w}^2 + ff_x\right), \tag{5.36a}$$

$$u_2 = -f - \boldsymbol{i}u_1 + \varepsilon(f_x - 2\boldsymbol{i}fu_0) + \varepsilon^2\bar{w}f. \tag{5.36b}$$

Now we expand the fields $\bar{w}(\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k w_k$ and $f(\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k f_k$, and plug the formal power series for $\bar{w}$ and f in previous formulas. Hence, starting from $w_0=u_{12}-\boldsymbol{i}u_{0;x}$ and $f_0=-u_2-\boldsymbol{i}u_1$, we obtain the recurrence relations between conserved densities w_k and auxiliary quantities f_k:

$$
\begin{aligned}
w_0 &= u_{12}-\boldsymbol{i}u_{0;x},\\
f_0 &= -u_2-\boldsymbol{i}u_1,\\
w_1 &= -2\boldsymbol{i}u_0w_0+D_xw_0+2\boldsymbol{i}f_0u_1,\\
f_1 &= D_xf_0-2\boldsymbol{i}u_0f_0,
\end{aligned}
$$

and also

$$
\begin{aligned}
w_n &= -2\boldsymbol{i}u_0w_{n-1}+D_xw_{n-1}+2\boldsymbol{i}f_{n-1}u_1+\sum_{k=0}^{n-2}(w_kw_{n-2-k}-f_kD_xf_{n-2-k}),\\
f_n &= D_xf_{n-1}-2\boldsymbol{i}u_0f_{n-1}+\sum_{k=0}^{n-2}w_kf_{n-2-k}.
\end{aligned}
$$

We note that $w_t\in\operatorname{im}\mathrm{d}/\mathrm{d}x$ and $f_t\notin\operatorname{im}\mathrm{d}/\mathrm{d}x$. This means that only the densities w_k are conserved and f_k are just auxiliary quantities (which are not conserved in general).

Now let us prove that all w_k are non-trivial (i.e., not contained in image of $\mathrm{d}/\mathrm{d}x$). Consider the zero-order component $\breve{w}_k^{(0,12)}(u_0,u_{12})$ of conserved densities $w_k([u_0,u_1,u_2,u_{12}])$ such that $\breve{w}_k^{(0,12)}(u_0,u_{12})$ depends only on u_0 and u_{12}. We have that

$$
\begin{aligned}
\breve{w}_0^{(0,12)} &= u_{12},\\
\breve{w}_1^{(0,12)} &= -2\boldsymbol{i}u_0u_{12},\\
\breve{w}_n^{(0,12)} &= -2\boldsymbol{i}u_0\breve{w}_{n-1}^{(0,12)}+\sum_{k=0}^{n-2}\breve{w}_k^{(0,12)}\breve{w}_{n-2-k}^{(0,12)}.
\end{aligned}
$$

It is readily seen that quantities $\breve{w}_n^{(0,12)}$ are in a form of $\breve{w}_n^{(0,12)}=u_{12}\cdot g_k$, where some functions $g_k\in C^\infty(\mathcal{E}^\infty)$ depend only on u_0 and u_{12}. Let us consider first term of quantities $\breve{w}_n^{(0,12)}$ which are linear in u_{12}. Only the term $-2iu_0\breve{w}_{n-1}^{(0,12)}$ in $\breve{w}_n^{(0,12)}$ is linear in u_{12} because the term $\breve{w}_k^{(0,12)}\breve{w}_{n-2-k}^{(0,12)}$ is quadratic in u_{12}. We obtain that $\breve{w}_n^{(0,12)}=(-2\boldsymbol{i}u_0)^nu_{12}+u_{12}^2\cdot(\ldots)$. This implies that $w_n=(-2\boldsymbol{i}u_0)^nu_{12}+\ldots$ and $w_n\notin\operatorname{im}\mathrm{d}/\mathrm{d}x$ because $(-2\boldsymbol{i}u_0)^nu_{12}\notin\operatorname{im}\mathrm{d}/\mathrm{d}x$. This proves that w_n is non-trivial for all $k\geqslant -1$.

Finally, let us obtain the generating function for the zero differential order component $\breve{w}(u_0,u_1,u_2,u_{12},\varepsilon)$ of the series $w([u_0,u_1,u_2,u_{12}],\varepsilon)$. For zero-order terms in (5.36) we have

$$
u_{12}=\breve{w}+\varepsilon(-2\boldsymbol{i}\breve{f}u_1+2\boldsymbol{i}u_0\breve{w})-\varepsilon^2\breve{w}^2, \tag{5.37}
$$

$$
u_2=-\breve{f}-\boldsymbol{i}u_1-2\varepsilon\boldsymbol{i}\breve{f}u_0+\varepsilon^2\breve{w}\breve{f}. \tag{5.38}
$$

Solving equation (5.38) with respect to $\breve{f}$ we obtain

$$\breve{f} = -\frac{u_2 + \boldsymbol{i}u_1}{1 + 2\varepsilon\boldsymbol{i}u_0 - \varepsilon^2\breve{w}}. \tag{5.39}$$

Substituting (5.39) for $\breve{f}$ in (5.37), we get

$$-\varepsilon^4\breve{w}^3 + 2\varepsilon^2(2\boldsymbol{i}\varepsilon u_0 + 1)\breve{w}^2 + (\varepsilon^2(4u_0^2 - u_{12}) - 4\boldsymbol{i}\varepsilon u_0 - 1)\breve{w} + u_{12} + 2\boldsymbol{i}\varepsilon(u_0u_{12} - u_1u_2) = 0.$$

In agreement with $\lim_{\varepsilon\to 0}\breve{w} = u_{12}$, we pick the root (5.35) for this third-order algebraic equation. $\square$

We finally remark that the reduction $u_0 = 0$, $u_1 = 0$ (and the change of notation $u_2 \to \xi$, $u_{12} \to u$) maps Gardner's deformation (5.33)-(5.34) for the N=2, a=4 SKdV equation to the Gardner's deformation for N=1 supersymmetric Korteweg–de Vries equation (5.23) (see Example 19 on p. 64).

Overall comment

By now the Gardner deformation problem for the N=2 supersymmetric a=4 Korteweg–de Vries equation ([99]) is solved. In this section we have found the solution which is an alternative to our previous result in Chapter 3. Namely, we introduced the nonlocal bosonic and fermionic variables in such a way that the rules to differentiate them are consistent by virtue of the super-equation at hand and second, the entire system retracts to the standard KdV equation and the classical Gardner deformation for it ([101]) under setting to zero the fermionic nonlocal variable and the first three components of the N=2 superfield in (1.2). At the same time, the structure under study is equivalent to the $\mathfrak{sl}(2 \mid 1)$-valued zero-curvature representation for this super-equation; the zero-curvature representation contains the non-removable spectral parameter, which manifests the integrability.

Our second solution of P. Mathieu's Open problem 2 (see [99]) relies on the interpretation of both Gardner's deformations and zero-curvature representations in similar terms, as a specific type of nonlocal structures over the equation of motion [62]. However, we emphasize that generally there is no one-to-one correspondence between the two constructions, so that the interpretation of deformations in the Lie-algebraic language is not always possible. Because this correlation between the two approaches to integrability was not revealed in the canonical formulation of the deformation problem [99], there appeared some attempts to solve it within the classical scheme but the progress was partial [4, 84]. Still, the use of zero-curvature representations in this context could have given the sought-for deformation long ago.

Let us notice also that projective substitution (5.5) correlates the super-dimension of the Lie algebra in a zero-curvature representation for a differential equation with the numbers of bosonic and fermionic nonlocalities over the same system: a subalgebra of $\mathfrak{gl}(p \mid q)$ yields at most $p-1$ bosonic and q fermionic variables. This implies that, for a covering over the $N{=}2$ supersymmetric KdV equation (1.2) to extend Gardner's deformation (2.1) in its classical sense $\mathfrak{m}_\varepsilon\colon \mathcal{E}_\varepsilon \to \mathcal{E}$ (see [59, 84, 101]), the extension $\mathcal{E}_\varepsilon$ must be the system of evolution equations upon two bosonic and two fermionic fields. Therefore, one may have to use the $\mathfrak{sl}(3 \mid 2)$-valued zero-curvature representations. This outlines the working approach to a yet another method of solving the Gardner deformation problem for the $N{=}2$ supersymmetric Korteweg–de Vries systems (1.2), which we leave as a new open problem.

Remark 10. In Theorem 4 on p. 62 we obtained a deformation for Krasil'shchik–Kersten system with one of the extended equations presented in non-divergent form (which is different from the classical definition of Gardner's deformation). In Theorem (6) on p. 70 we have made another deviation from the classical construction of Gardner's deformations, namely, the number of extended equations is not equal to the number of original equations. However, both of these deformations yield recurrence relations between conserved densities, whence one could call them "semi-classical" Gardner's deformations.

5.4 Families of coverings and the Frölicher–Nijenhuis bracket

Consider a $(k_0|k_1)$-dimensional covering $\tau\colon \tilde{\mathcal{E}} = W \times \mathcal{E}^\infty \to \mathcal{E}^\infty$ with even nonlocal coordinates $w^1, \ldots, w^{k_0}$ and odd nonlocal coordinates $f^1, \ldots, f^{k_1}$ on a $(k_0|k_1)$-dimensional auxiliary supermanifold W. The prolongations $\tilde{D}_{x^i}$ of the total derivatives $\bar{D}_{x^i}$ to the covering equation $\tilde{\mathcal{E}}$ are given by the formulas [12, 78]

$$\tilde{D}_{x^i} = \bar{D}_{x^i} + w^p_{x^i}\frac{\partial}{\partial w^p} + f^q_{x^i}\frac{\vec{\partial}}{\partial f^q}, \qquad 1 \leqslant i \leqslant n.$$

These total derivatives $\tilde{D}_{x^i}$ determine the Cartan distribution $\mathcal{C}(\tilde{\mathcal{E}})$ on the covering equation $\tilde{\mathcal{E}}$. In turn, the Cartan distribution $\mathcal{C}(\tilde{\mathcal{E}})$ yields the connection $\mathcal{C}_{\tilde{\mathcal{E}}}\colon\ \mathrm{D}(M) \to \mathrm{D}(\tilde{\mathcal{E}})$; the corresponding connection form $U_{\tilde{\mathcal{E}}} \in \mathrm{D}(\Lambda^1(\tilde{\mathcal{E}}))$ is the *structural element* of the covering τ. Expressing $U_{\tilde{\mathcal{E}}}$ in coordinates, we obtain

$$U_{\tilde{\mathcal{E}}} = \bar{\mathrm{d}}_{\mathcal{C}}(u^k_{\sigma_{\bar{0}}})\frac{\partial}{\partial u^k_{\sigma_{\bar{0}}}} + \bar{\mathrm{d}}_{\mathcal{C}}(\xi^a_{\sigma_{\bar{1}}})\frac{\vec{\partial}}{\partial \xi^a_{\sigma_{\bar{1}}}} + (\mathrm{d}w^p - w^p_{x^i}\,\mathrm{d}x^i)\frac{\partial}{\partial w^p} + (\mathrm{d}f^q - f^q_{x^i}\,\mathrm{d}x^i)\frac{\vec{\partial}}{\partial f^q}.$$

Next, let us recall that the Frölicher–Nijenhuis bracket $[\cdot,\cdot]^{\mathrm{FN}}$ on $\mathrm{D}(\Lambda^*(\tilde{\mathcal{E}}))$ is defined by the formula [78]

$$[\Omega, \Theta]^{\mathrm{FN}}(g) = \mathrm{L}_\Omega(\Theta(g)) - (-1)^{rs+\mathsf{p}(\Omega)\mathsf{p}(\Theta)}\,\mathrm{L}_\Theta(\Omega(g)),$$

where $\Omega \in \mathrm{D}(\Lambda^r(\tilde{\mathcal{E}}))$, $\Theta \in \mathrm{D}(\Lambda^s(\tilde{\mathcal{E}}))$, and $g \in C^\infty(\tilde{\mathcal{E}})$; here $\mathrm{L}_\Omega = \mathrm{i}_\Omega \circ \mathrm{d} + \mathrm{d} \circ \mathrm{i}_\Omega$ is the Lie derivative.

Let $\tau_\lambda \colon \tilde{\mathcal{E}}_\lambda = W_\lambda \times \mathcal{E}^\infty \to \mathcal{E}^\infty$ be a smooth family of coverings over $\mathcal{E}^\infty$ depending on a parameter $\lambda \in \mathbb{C}$ and U_λ be the corresponding characteristic element of τ_λ. Following [50], we assume that the distributions $\mathcal{C}(\tilde{\mathcal{E}}_\lambda)$ are diffeomorphic to each other at different values of λ under a smooth family of diffeomorphims of the manifolds $\tilde{\mathcal{E}}_\lambda$. The evolution of U_λ with respect to λ is described by the equation [49, 50]

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} U_\lambda = [X, U_\lambda]^{\mathrm{FN}}, \tag{5.40}$$

where $X \in \mathrm{D}(\tilde{\mathcal{E}})$ is some vector field on $\tilde{\mathcal{E}}_\lambda$.

Example 20. Let us consider the $N{=}2$, $a{=}4$ SKdV equation (3.5) and a family of coverings over it derived from the zero-curvature representation which we addressed in Example 9. We now solve equation (5.40) in three steps.

We begin with the covering derived from the Gardner deformation [101],

$$\begin{aligned} w_x &= \tfrac{1}{\varepsilon}(w - u_{12}) - \varepsilon w^2, && (5.8a)\\ w_t &= \tfrac{1}{\varepsilon}(u_{12;xx} + 2u_{12}^2) + \tfrac{1}{\varepsilon^2}u_{12;x} + \tfrac{1}{\varepsilon^3}u_{12} + \big(-2u_{12;x} - \tfrac{2}{\varepsilon}u_{12} - \tfrac{1}{\varepsilon^3}\big)\, w + \big(2\varepsilon u_{12} + \tfrac{1}{\varepsilon}\big)\, w^2, && (5.8b) \end{aligned}$$

of the Korteweg–de Vries equation

$$u_{12;t} = -u_{12;xxx} - 6u_{12}u_{12;x}. \tag{1.1}$$

The solution of equation (5.40) for Gardner's deformation (5.45) of KdV equation (1.1) is

$$\begin{aligned} X_1 &= \varepsilon^{-2}(-x\,\partial/\partial x - 3t\,\partial/\partial t + 2u_{12}\,\partial/\partial u_{12} + \ldots + 2w\,\partial/\partial w),\\ X_2 &= -2\varepsilon(6t\,\partial/\partial x + \partial/\partial u_{12} + \ldots) - \partial/\partial w. \end{aligned}$$

We recall from [114] that Sasaki used the scaling symmetry (encoded by the local part of the vector field X_1, i.e., without $2\varepsilon^{-2}w\,\partial/\partial w$) for eliminating the parameter ε (see Section 5.5; we refer to diagram (5.50) on p. 82 in particular).

Second, let us consider the Kaup–Boussinesq equation [18, 54],

$$u_{0;s} = (-u_{12} + 2u_0^2)_x, \qquad u_{12;s} = (u_{0;xx} + 4u_0u_{12})_x,$$

and take its higher symmetry

$$\begin{aligned} u_{0;t} &= -u_{0;xxx} + \big(4u_0^3 - 6u_0u_{12}\big)_x, && (5.42a)\\ u_{12;t} &= -u_{12;xxx} - 6u_{12}u_{12;x} + 12u_{0;x}u_{0;xx} + 6u_0u_{0;xxx} + 12\big(u_0^2u_{12}\big)_x. && (5.42b) \end{aligned}$$

We recall that system (5.42) is the bosonic limit of (3.5) with $a = 4$ under setting $u_1 = u_2 = 0$.

A family of coverings over equation (5.42) is determined by the formulas[6]

$$\begin{aligned}
\underline{w_x} &= \underline{-\varepsilon w^2} + \varepsilon^{-1}(\underline{w - u_{12}} - u_0^2) + \boldsymbol{i}\varepsilon^{-2}u_0,\\
\underline{w_t} &= 2\varepsilon w^2(-2u_0^2 + \underline{u_{12}}) + 2w(-\boldsymbol{i}wu_0 + 4u_0u_{0;x}\underline{-u_{12;x}}) + \varepsilon^{-1}(\underline{w^2 - 2wu_{12} + 2u_{12}^2}\\
&\quad \underline{+u_{12;xx}} + 2\boldsymbol{i}wu_{0;x} - 4u_0^4 - 2u_0^2u_{12} - 4u_0u_{0;xx} + 4wu_0^2 - 4u_{0;x}^2) + \varepsilon^{-2}(2\boldsymbol{i}wu_0\\
&\quad + 2\boldsymbol{i}u_0^3 - 4\boldsymbol{i}u_0u_{12} - 4u_0u_{0;x} - \boldsymbol{i}u_{0;xx} + \underline{u_{12;x}}) + \varepsilon^{-3}(\underline{u_{12} - w} - u_0^2 - \boldsymbol{i}u_{0;x}) - \boldsymbol{i}\varepsilon^{-4}u_0.
\end{aligned}$$

At every ε, such coverings are obtained by the standard change of Lie algebra's realization in a zero-curvature representation for (5.42); in turn, that representation can be derived by using the reduction $u_1 = u_2 = 0$ in the zero-curvature representation for the N=2, a=4 SKdV equation (3.5) (see [28] and Example 9). Remarkably, this zero-curvature representation for (5.42) was re-discovered in [14] not in the context of super-system (3.5).

For this family of coverings over system (5.42), the solution of equation (5.40) is given by the vector field

$$X = \varepsilon^{-1}(-x\,\partial/\partial x - 3t\,\partial/\partial t + u_0\,\partial/\partial u_0 + 2u_{12}\,\partial/\partial u_{12} + \ldots + 2w\,\partial/\partial w)$$

We note that, the same as it is in the case of KdV equation (1.1), we obtain the vector field corresponding to the scaling symmetry.

Finally, let us consider the full N=2, a=4 SKdV equation (3.5) and the (1|1)-dimensional covering (5.30)-(5.31) over it . We find that the solution of equation (5.40) for this covering is the vector field

$$\begin{aligned}
X = \varepsilon^{-1}(-x\,\partial/\partial x - 3t\,\partial/\partial t + u_0\,\partial/\partial u_0 + \tfrac{3}{2}u_1\,\partial/\partial u_1 + \tfrac{3}{2}u_2\,\partial/\partial u_2\\
+ 2u_{12}\,\partial/\partial u_{12} + \cdots + 2w\,\partial/\partial w + \tfrac{3}{2}f\,\partial/\partial f).
\end{aligned}$$

It has been obtained by solving equation (5.40) explicitly using the analytic software [73].

We note again that – as we had it in the above two reductions of the N=2, a=4-SKdV – we obtain the vector filed corresponding to the scaling symmetry of the underlying equation.[7]

[6] Here and in what follows we underline the covering that encodes Gardner's deformation (5.41) for the classical KdV equation (1.1).

[7] For scaling-invariant families of coverings depending on a parameter of non-zero homogeneity weight, one could always try to find a solution of equation (5.40) by taking the scaling symmetry of the underlying PDE.

Let us consider two representation of $\mathfrak{g}$:

1. $\rho\colon \mathfrak{g} \to \mathrm{Mat}(k_0+1, k_1)$, that is, a matrix representation;
2. $\varrho\colon \mathfrak{g} \to \mathrm{Vect}(W; \mathrm{poly})$, which is the representation in the space of vector fields with polynomial coefficients on the $(k_0|k_1)$-dimensional supermanifold W with local parity-even coordinates $w^1, \ldots, w^{k_0}$ and $f^1, \ldots, f^{k_1}$ of odd parity.

Let $\alpha = a^i \rho(e_i)\,\mathrm{d}x + b^j \rho(e_j)\,\mathrm{d}t$ be a $\mathfrak{g}$-valued zero-curvature representations for the system $\mathcal{E}$. Construct a one-dimensional covering with nonlocal variable w over $\mathcal{E}^\infty$ such that

$$w_x = -a^i \varrho(e_i) \lrcorner \mathrm{d}w, \tag{5.43a}$$
$$w_t = -b^j \varrho(e_j) \lrcorner \mathrm{d}w. \tag{5.43b}$$

Consider two mappings, $\boldsymbol{\partial}_\alpha = \bar{\mathrm{d}}_h - [\alpha, \cdot]\colon \mathfrak{g} \otimes \Lambda^0(\mathcal{E}^\infty) \to \mathfrak{g} \otimes \Lambda^1(\mathcal{E}^\infty)$ (see [94])and $\partial_U = [\cdot, U_\lambda]^{\mathrm{FN}}\colon\ \mathrm{D}(\Lambda^0(\tilde{\mathcal{E}})) \to \mathrm{D}(\Lambda^1(\tilde{\mathcal{E}}))$ (see [50]). We recall that the mappings $\boldsymbol{\partial}_\alpha$ and ∂_U yield the horizontal [94] and Cartan [50] cohomologies, respectively. However, we claim that in the geometry at hand one of these two differentials is a particular instance of the other by virtue of the switch $\rho \rightleftarrows \varrho$ between the Lie superalgebra's representations.

Lemma 2. The following diagram is commutative:

$$\begin{array}{ccccc}
\mathfrak{g} \otimes \Lambda^0(\mathcal{E}^\infty) & \xrightarrow{\rho} & \mathrm{Mat}(k_0+1|k_1) \otimes \Lambda^0(\mathcal{E}^\infty) & \xrightarrow{\boldsymbol{\partial}_\alpha} & \mathrm{Mat}(k_0+1|k_1) \otimes \Lambda^1(\mathcal{E}^\infty) \\
\Big\| & & \Big\downarrow{\scriptstyle\nabla} & & \Big\downarrow{\scriptstyle\nabla} \\
\mathfrak{g} \otimes \Lambda^0(\mathcal{E}^\infty) & \xrightarrow{\varrho} & \mathrm{D}(\Lambda^0(\tilde{\mathcal{E}})) & \xrightarrow{\partial_U} & \mathrm{D}(\Lambda^1(\tilde{\mathcal{E}})),
\end{array}$$

where $\nabla = \varrho \circ \rho^{-1}$ is a switch from the representation ρ to the representation ϱ for the Lie superalgebra $\mathfrak{g}$.

Proof. Consider $\gamma = q^k \cdot e_k \in \mathfrak{g} \otimes \Lambda^0(\mathcal{E}^\infty)$ and put $\varrho(\gamma) = X \in \mathrm{D}(\Lambda^0(\tilde{\mathcal{E}}))$ and $\rho(\gamma) = Q \in \mathrm{Mat}(k_0+1|k_1) \otimes \Lambda^0(\mathcal{E}^\infty)$. A direct calculation shows that

$$\begin{aligned}
&(\nabla \circ \boldsymbol{\partial}_\alpha \circ \rho)(\gamma) = (\nabla \circ \boldsymbol{\partial}_\alpha)(Q) = \nabla(\bar{\mathrm{d}}_h Q - [\alpha, Q]) \\
&\quad = \nabla\Big(\mathrm{d}x\,(\bar{D}_x(q^k)\rho(e_k) - [a^i\rho(e_i), q^k\rho(e_k)]) + \mathrm{d}t\,(\bar{D}_t(q^k)\rho(e_k) - [b^j\rho(e_j), q^k\rho(e_k)]\Big) \\
&\quad = \mathrm{d}x\,\big(\bar{D}_x(q^k)\varrho(e_k) + [-a^i\varrho(e_i), q^k\varrho(e_k)]\big) + \mathrm{d}t\,(\bar{D}_t(q^k)\varrho(e_k) + [-b^i\varrho(e_i), q^k\varrho(e_k)]).
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
&(\partial_U \circ \varrho)(\gamma) = \partial_U X = [X, U_\lambda]^{\mathrm{FN}} \\
&\quad = \left[\mathrm{d}x\left(\tilde{D}_x(X \lrcorner \mathrm{d}w) - (X \lrcorner \mathrm{d}w)\frac{\partial w_x}{\partial w}\right) + \mathrm{d}t\left(\tilde{D}_t(X \lrcorner \mathrm{d}w) - (X \lrcorner \mathrm{d}w)\frac{\partial w_t}{\partial w}\right)\right] \otimes \frac{\partial}{\partial w}.
\end{aligned}$$

By using the formula $\tilde{D}_x(X \lrcorner \, \mathrm{d}w) = \bar{D}_x(X \lrcorner \, \mathrm{d}w) + w_x \frac{\partial}{\partial w}(X \lrcorner \, \mathrm{d}w)$, we continue the equality and obtain that

$$
\begin{aligned}
&= \Big[\mathrm{d}x \left(\bar{D}_x(X \lrcorner \, \mathrm{d}w) + w_x \frac{\partial}{\partial w}(X \lrcorner \, \mathrm{d}w) - (X \lrcorner \, \mathrm{d}w) \frac{\partial w_x}{\partial w} \right) \\
&\quad + \mathrm{d}t \left(\bar{D}_t(X \lrcorner \, \mathrm{d}w) + w_t \frac{\partial}{\partial w}(X \lrcorner \, \mathrm{d}w) - (X \lrcorner \, \mathrm{d}w) \frac{\partial w_t}{\partial w} \right) \Big] \otimes \frac{\partial}{\partial w} \\
&\Big[\mathrm{d}x \left(\bar{D}_x(X) \lrcorner \, \mathrm{d}w + [\![w_x \frac{\partial}{\partial w}, X]\!] \lrcorner \, \mathrm{d}w \right) + \mathrm{d}t \left(\bar{D}_t(X) \lrcorner \, \mathrm{d}w + [\![w_t \frac{\partial}{\partial w}, X]\!] \lrcorner \, \mathrm{d}w \right) \Big] \otimes \frac{\partial}{\partial w}.
\end{aligned}
$$

From formulas (5.43) we infer that

$$
\begin{aligned}
&= \Big[\mathrm{d}x(\bar{D}_x(q^k)\varrho(e_k) + [-a^i \varrho(e_i), q^k \varrho(e_k)]) \lrcorner \, \mathrm{d}w \\
&\quad + \mathrm{d}t(\bar{D}_t(q^k)\varrho(e_k) + [-b^i \varrho(e_i), q^k \varrho(e_k)]) \lrcorner \, \mathrm{d}w) \Big] \otimes \frac{\partial}{\partial w} \\
&= \mathrm{d}x \left(\bar{D}_x(q^k)\varrho(e_k) + [-a^i \varrho(e_i), q^k \varrho(e_k)] \right) + \mathrm{d}t(\bar{D}_t(q^k)\varrho(e_k) + [-b^i \varrho(e_i), q^k \varrho(e_k)]).
\end{aligned}
$$

We finally obtain that $(\nabla \circ \boldsymbol{\partial}_\alpha \circ \rho)(\gamma) = (\partial_U \circ \varrho)(\gamma)$, which proves our claim. $\square$

Now let us study in more detail the case of *removable* parameters. Let $\alpha(\lambda) = a^i \rho(e_i) \, \mathrm{d}x + b^j \rho(e_j) \, \mathrm{d}t$ be a smooth family of $\mathfrak{g}$-valued zero-curvature representations for the system $\mathcal{E}$ but let the parameter $\lambda \in \mathbb{C}$ be removable. By Proposition 5, there is a $\mathfrak{g}$-matrix $Q = q^k \rho(e_k)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\alpha = \bar{\mathrm{d}}_h Q - [\alpha, Q].$$

In components, we have that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}(a^i)\rho(e_i) &= \bar{D}_x(q^k)\rho(e_k) - a^i q^k \rho([e_i, e_k]), \\
\frac{\mathrm{d}}{\mathrm{d}\lambda}(b^j)\rho(e_i) &= \bar{D}_t(q^k)\rho(e_k) - b^j q^k \rho([e_j, e_k]).
\end{aligned}
$$

By virtue of the representation ϱ, at every λ the $\mathfrak{g}$-matrix $Q = q^k \rho(e_k)$ determines the vector field $X = q^k \varrho(e_k)$ on $\tilde{\mathcal{E}}$.

The following proposition is a regular generator of solutions for equation (5.40) in the case of coverings derived from zero-curvature representations with removable parameters.[8]

Proposition 7. The vector field $X = q^k \varrho(e_k)$ satisfies structure equation (5.40).

[8] It was remarked in [49] that the formalism of zero-curvature representations and their parametric families can be viewed as a special case of the Frölicher–Nijenhuis bracket formalism for deformations of coverings of unspecified nature; we thus substantiate that claim from *loc. cit.* by giving an explicit proof.

Proof. From Lemma 2 we infer that

$$[X, U_\lambda]^{\mathrm{FN}} = \Big[\mathrm{d}x\,\big(\bar{D}_x(q^k)\varrho(e_k) + [-a^i\varrho(e_i), q^k\varrho(e_k)]\big) \lrcorner\, \mathrm{d}w \\ + \mathrm{d}t(\bar{D}_t\,\big(q^k)\varrho(e_k) + [-b^i\varrho(e_i), q^k\varrho(e_k)]\big) \lrcorner\, \mathrm{d}w\Big] \otimes \frac{\partial}{\partial w}.$$

Using (5.44), we obtain that

$$= \Big[\mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}\lambda}(a^i)(\varrho(e_i) \lrcorner\, \mathrm{d}w) + \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}\lambda}(b^i)(\varrho(e_i) \lrcorner\, \mathrm{d}w)\Big] \otimes \frac{\partial}{\partial w} \\ = \Big[-\frac{\mathrm{d}}{\mathrm{d}\lambda}(w_x)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}\lambda}(w_t)\,\mathrm{d}t\Big] \otimes \frac{\partial}{\partial w} = \frac{\mathrm{d}}{\mathrm{d}\lambda}U_\lambda.$$

This proves that the vector field X is a solution of equation (5.40). □

Remark 11. This proof can be easily extended to the case of any finite n and k_0, $k_1 < \infty$.

Example 21. Let us illustrate the claim of Proposition 7. Namely, let us construct a $(1|1)$-dimensional covering over the N=2, a=4 SKdV equation (3.5) by taking the $\mathfrak{sl}(2|1)$-valued zero-curvature representation β from Example 10 on p. 48. Using representation ϱ from Example 14, we obtain

$$\begin{aligned}
w_x &= \lambda^2 + 2\lambda w + w^2 + u_0^2 + u_{12} - f_2u_2 + \boldsymbol{i}f_2u_1,\\
f_x &= \lambda f_2 + f_2w + \boldsymbol{i}f_2u_0 + u_2 + \boldsymbol{i}u_1,\\
w_t &= 2\lambda^2(2u_0^2 - u_{12}) + \lambda(8wu_0^2 - 4wu_{12} + 8u_0u_{0;x} - 2u_{12;x} + fu_{2;x} - \boldsymbol{i}fu_{1;x}\\
&\quad + 4\boldsymbol{i}fu_2u_0 + 4fu_1u_0) + 4w^2u_0^2 - 2w^2u_{12} + 8wu_0u_{0;x} - 2wu_{12;x} + 4u_0^4 + 2u_0^2u_{12}\\
&\quad + 4u_0u_{0;xx} - 2u_{12}^2 + 4u_{0;x}^2 - u_{12;xx} + fwu_{2;x} - \boldsymbol{i}fwu_{1;x} + 5\boldsymbol{i}fu_0u_{2;x} + 5fu_0u_{1;x}\\
&\quad + fu_{2;xx} - \boldsymbol{i}fu_{1;xx} + 4\boldsymbol{i}fu_2wu_0 - 8fu_2u_0^2 + 2fu_2u_{12} + 4\boldsymbol{i}fu_2u_{0;x} + 4fu_1wu_0\\
&\quad + 8\boldsymbol{i}fu_1u_0^2 - 2\boldsymbol{i}fu_1u_{12} + 4fu_1u_{0;x} + u_2u_{2;x} + 8u_2u_1u_0 + u_1u_{1;x},\\
f_t &= \lambda(-u_{2;x} - \boldsymbol{i}u_{1;x} + 4fu_0^2 - 2fu_{12} + 4\boldsymbol{i}u_2u_0 - 4u_1u_0) - wu_{2;x} - \boldsymbol{i}wu_{1;x} + 5\boldsymbol{i}u_0u_{2;x}\\
&\quad - 5u_0u_{1;x} - u_{2;xx} - \boldsymbol{i}u_{1;xx} + 4fwu_0^2 - 2fwu_{12} + 4\boldsymbol{i}fu_0^3 - 6\boldsymbol{i}fu_0u_{12} + 4fu_0u_{0;x}\\
&\quad - fu_{0;xx}\boldsymbol{i} - fu_{12;x} - 4fu_2u_1\boldsymbol{i} + 4u_2wu_0\boldsymbol{i} + 8u_2u_0^2 - 2u_2u_{12} + 4\boldsymbol{i}u_2u_{0;x} - 4u_1wu_0\\
&\quad + 8\boldsymbol{i}u_1u_0^2 - 2\boldsymbol{i}u_1u_{12} - 4u_1u_{0;x}.
\end{aligned}$$

In agreement with Proposition 7, we find the solution $X = \partial/\partial w$ of equation (5.40): indeed, this field is obtained from the $\mathfrak{sl}(2|1)$-matrix Q which we introduced in Example 10.

We confirmed that a switch between the representations of Lie (super)algebras establishes a link between the two classes of nonlocal geometries and also between the arising differentials. In particular, by analyzing this relation in the case of zero-curvature representations with removable parameters λ, we explicitly described the equivalence classes of τ_λ-shadows that determine, by virtue of structure equation (5.40), the evolution of Cartan's structural elements in families of coverings τ_λ.

5.5 Two descriptions of one elimination procedure: an example

We now analyze the following tautological construction: by re-addressing Sasaki,[9] see [114], we first track how the scaling symmetry of KdV equation (1.1) acts on its standard matrix Lax pair; on the other hand, we reveal how these objects are phrased in the language of coverings.

Recall that the Korteweg–de Vries equation is

$$\mathcal{E} = \{u_t = -u_{xxx} - 6uu_x\}\,. \tag{1.1}$$

Consider the family of coverings $\tau_\eta\colon \tilde{\mathcal{E}}_\eta \to \mathcal{E}$ over it,

$$v_x = 2v\eta - (v^2 + u), \tag{5.45a}$$
$$v_t = -8\eta^3 v + 4\eta^2(v^2+u) + 2\eta(-2vu + u_x) + 2v^2u - 2vu_x + 2u^2 + u_{xx}; \tag{5.45b}$$

these formulas are obtained from the following $\mathfrak{sl}_2$-valued zero-curvature representation (see [114]),

$$\alpha_\eta = \begin{pmatrix} \eta & u \\ -1 & -\eta \end{pmatrix} \mathrm{d}x + \begin{pmatrix} -(4\eta^3 + 2\eta u + u_x) & -(u_{xx} + 2\eta u_x + 4\eta^2 u + 2u^2) \\ 4\eta^2 + 2u & 4\eta^3 + 2\eta u + u_x \end{pmatrix} \mathrm{d}t.$$

Let us recall that the parameter η can not be removed from the zero-curvature representations α_η by using gauge transformations. However, it can be eliminated by using a wider class of transformations. Namely, consider the scaling symmetry of equation (1.1),

$$x \mapsto \eta x, \quad t \mapsto \eta^3 t, \quad u \mapsto \eta^{-2} u, \qquad \eta \in \mathbb{R}.$$

Using it, one transforms the zero-curvature representation α_η into

$$\alpha'_\eta = \begin{pmatrix} 1 & \eta u \\ -\eta^{-1} & -1 \end{pmatrix} \mathrm{d}x + \begin{pmatrix} -(4 + 2u + u_x) & -\eta(u_{xx} + 2u_x + 4u + 2u^2) \\ \eta^{-1}(4+2u) & 4 + 2u + u_x \end{pmatrix} \mathrm{d}t.$$

[9] A parameter-dependent zero-curvature representation for Burgers' equation was considered in [23] in the same context of pseudospherical surfaces as in Sasaki's paper [114]. We refer to [94] for analysis of removability of the parameter in that zero-curvature representation for Burgers' equation [23].

The parameter η in α'_η is removable under the gauge transformation

$$g = \begin{pmatrix} \eta^{-1/2} & 0 \\ 0 & \eta^{1/2} \end{pmatrix} \in C^\infty(\mathcal{E}^\infty, GL_2(\mathbb{C})),$$

that is, we have that $(\alpha'_\eta)^g = \alpha'_\eta\big|_{\eta=1} = \alpha_\eta\big|_{\eta=1}$.

Let us now address the removability of parameter η in coverings (5.45) in terms of the formalism of Cartan's structural element.

For a vector field

$$X = a \otimes \frac{\partial}{\partial x} + b \otimes \frac{\partial}{\partial t} + \omega_\sigma \otimes \frac{\partial}{\partial u_\sigma} + \varphi \otimes \frac{\partial}{\partial v},$$

the equation for evolution of Cartan's structural element,

$$\frac{\mathrm{d}}{\mathrm{d}\eta} U_\eta = [X, U_\eta]^{\mathrm{FN}}, \tag{5.40}$$

splits into the system

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}\eta} v_x &= \tilde{D}_x \varphi - \varphi \frac{\partial v_x}{\partial v} - \omega_\sigma \frac{\partial v_x}{\partial u_\sigma} + b\left(\frac{\partial v_x}{\partial u_\sigma} u_{\sigma t} + \frac{\partial v_x}{\partial v} v_t - \tilde{D}_x v_t\right) - v_t \frac{\partial b}{x} \\ &\quad + a\left(-\tilde{D}_x v_x + \frac{\partial v_x}{\partial u_\sigma} u_{\sigma x} + \frac{\partial v_x}{\partial v} v_x\right) - v_x \frac{\partial a}{\partial x}, \qquad (5.46a) \\ -\frac{\mathrm{d}}{\mathrm{d}\eta} v_t &= \tilde{D}_t \varphi - \varphi \frac{\partial v_t}{\partial v} - \omega_\sigma \frac{\partial v_t}{\partial u_\sigma} + b\left(\frac{\partial v_t}{\partial u_\sigma} u_{\sigma t} + \frac{\partial v_t}{\partial v} v_t - \tilde{D}_t v_t\right) - v_t \frac{\partial b}{\partial t} \\ &\quad + a\left(-\tilde{D}_t v_x + \frac{\partial v_t}{\partial u_\sigma} u_{\sigma x} + \frac{\partial v_t}{\partial v} v_x\right) - v_x \frac{\partial a}{\partial t}, \qquad (5.46b) \\ \omega_{\sigma x} &= \tilde{D}_x \omega_\sigma - u_{\sigma t} \frac{\partial b}{\partial x} - u_{\sigma x} \frac{\partial a}{\partial x}, \qquad (5.46c) \\ \omega_{\sigma t} &= \tilde{D}_t \omega_\sigma - u_{\sigma t} \frac{\partial b}{\partial t} - u_{\sigma x} \frac{\partial a}{\partial t}. \qquad (5.46d) \end{aligned}$$

Suppose now that the vector field is vertical: $X^{\mathrm{v}} = \omega^{\mathrm{v}}_\sigma \otimes \partial/\partial u_\sigma + \varphi^{\mathrm{v}} \otimes \partial/\partial v$. This simplifies equation (5.46); it then becomes

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}\eta} v_x &= \tilde{D}_x \varphi^{\mathrm{v}} - \varphi^{\mathrm{v}} \frac{\partial v_x}{\partial v} - \omega^{\mathrm{v}}_\sigma \frac{\partial v_x}{\partial u_\sigma}, \qquad (5.47a) \\ -\frac{\mathrm{d}}{\mathrm{d}\eta} v_t &= \tilde{D}_t \varphi^{\mathrm{v}} - \varphi^{\mathrm{v}} \frac{\partial v_t}{\partial v} - \omega^{\mathrm{v}}_\sigma \frac{\partial v_t}{\partial u_\sigma}, \qquad (5.47b) \\ \omega^{\mathrm{v}}_{\sigma x} &= \tilde{D}_x \omega^{\mathrm{v}}_\sigma, \qquad (5.47c) \\ \omega^{\mathrm{v}}_{\sigma t} &= \tilde{D}_t \omega^{\mathrm{v}}_\sigma. \qquad (5.47d) \end{aligned}$$

Let us use the Ansatz

$$\omega^{\mathrm{v}} = \omega - au_x - bu_t, \quad \varphi^{\mathrm{v}} = \varphi - av_x - bu_t,$$

assuming that $a = a(x, t, \eta)$, $b = b(x, t, \eta)$, $\varphi = \varphi(\eta, u, v)$, and $\omega = \omega(\eta, u, v, u_x, u_{xx})$. By construction, the unknowns ω^{v} and φ^{v} satisfy system (5.47). Using the analytic software `Jets` [96] and `Crack` [73], we find the solution

$$\begin{aligned}
a &= 24c_4t\eta^3 + 2c_4x\eta + \tfrac{1}{\eta}(c_6 + x),\\
b &= 6c_4t\eta + \tfrac{1}{\eta}(-c_7 + 3t),\\
\omega &= 4c_4\eta^3 - 4c_4u\eta + u_xc_4 + \tfrac{1}{\eta}(-\tfrac{1}{2}u_xc_3 - 2u) + \tfrac{1}{2\eta^2}u_x,\\
\varphi &= 2c_4\eta^2 - c_4v^2 - c_3v - c_4u + \tfrac{c_3}{2\eta}(v^2 + u) - \tfrac{1}{2\eta^2}(v^2 + u),
\end{aligned}$$

which contains four arbitrary constants c_3, c_4, c_6, and c_7.

Let us set $c_3 = 0$, $c_4 = -1/(2\eta^2)$ at $\eta \neq 0$, $c_6 = 0$, and $c_7 = 0$. This determines the solution which corresponds to the lift of Galilean symmetry of (1.1):

$$X_2 = -2\eta(6t\,\partial/\partial x + \partial/\partial u + \ldots) - \partial/\partial v.$$

On the other hand, set $c_3 = 1/\eta$ if $\eta \neq 0$ and let $c_4 = 0$, $c_6 = 0$, and $c_7 = 0$. This yields the solution which corresponds to the lift of scaling symmetry of (1.1); namely, we have that

$$X_1 = \eta^{-2}(-x\,\partial/\partial x - 3t\,\partial/\partial t + 2u\,\partial/\partial u + \ldots + v\,\partial/\partial v). \tag{5.48}$$

The exponent of vector field (5.48) induces the transformation

$$x \mapsto \eta x, \quad t \mapsto \eta^3 t, \quad u \mapsto \eta^{-2}u, \quad v \mapsto \eta^{-1}v. \tag{5.49}$$

Its action on the covering τ_η in (5.45) results in the covering $\tau' = \tau_\eta\big|_{\eta=1}$, which is described by the formulas

$$\begin{aligned}
v_x &= 2v - (v^2 + u),\\
v_t &= -8v + 4v^2 + 4u - 4vu + 2u_x + 2v^2u - 2vu_x + 2u^2 + u_{xx}.
\end{aligned}$$

We claim that the covering τ' is the image of zero-curvature representation $(\alpha'_\eta)^g$ under a swapping of representations for the Lie algebra at hand. This is shown in the following diagram:

$$\begin{tikzcd}
\alpha_\eta \arrow[r, "\text{scaling}"] \arrow[d, equal] & \alpha'_\eta \arrow[r, "g"] & \alpha'_\eta\big|_{\eta=1} \arrow[d, "\nabla"] \\
\alpha_\eta \arrow[r, "\nabla"] & \tau \arrow[r, "(5.49)"] & \tau'.
\end{tikzcd} \tag{5.50}$$

We conclude that the problem of finding transformations (which are possibly not gauge) that eliminate the parameter in a given family of zero-curvature representations can be approached via a solution of equation (5.40) in the family of coverings which are the $(\rho \rightleftarrows \varrho)$-avatars of those zero-curvature representations.

Depending on their elimination scenario, "removable" parameters in zero-curvature representations are classified as follows:

1. First, there are parameters which are removable under gauge transformations (see [94, 97] by Marvan and [112, 113] by Sakovich).

2. There are parameters which can not be removed by using gauge transformations but which indicate the presence of conserved currents in zero-curvature representations and the reducibility of such representations,[10] (see [95] and [62, § 12]).

3. Thirdly, there are parameters which vanish under the action of those symmetries of the underlying differential equation which can not be lifted to the covering Maurer–Cartan equation (see [88, 114]).

4. Finally, there are parameters which can be eliminated by the same procedure as in the previous case but by using *shadows* of nonlocal symmetries in some auxiliary covering over the equation at hand (namely, *not* in the covering which grasps the ZCR geometry but in an extension of the equation's geometry by a set of "nonlocalities"), see [24, 25, 26].

[10] For example, consider a "fake" $\mathfrak{sl}_2$-valued zero-curvature representation $\alpha = \begin{pmatrix} 0 & X_1 + \lambda X_2 \\ 0 & 0 \end{pmatrix} \mathrm{d}x + \begin{pmatrix} 0 & T_1 + \lambda T_2 \\ 0 & 0 \end{pmatrix} \mathrm{d}t$ for an equation $\mathcal{E}$ possessing two conserved currents $D_t X_i = D_x T_i$, here $i = 1, 2$.

Chapter 6

Non-Abelian variational Lie algebroids

In this chapter we show that zero-curvature representations for PDE give rise to a natural class of non-Abelian variational Lie algebroids. We list all the components of such structures (cf. [70]); in particular, we show that Marvan's operator $\boldsymbol{\partial}_\alpha$ is the anchor. In section 6.1, non-Abelian variational Lie algebroids are realized via BRST-like homological evolutionary vector fields Q on superbundles à la [9]. Having enlarged the BRST-type setup to a geometry which goes in a complete parallel with the standard BV-zoo ([7, 8], see also [3]), in section 6.2 we extend the vector field Q to the evolutionary derivation $\widehat{Q}(\cdot) \cong [\![\widehat{S}, \cdot]\!]$ whose Hamiltonian functional $\widehat{S}$ satisfies the classical master-equation $[\![\widehat{S}, \widehat{S}]\!] = 0$.

In the earlier work [70] by Kiselev and van de Leur, classical notion of Lie algebroids [120] was upgraded from ordinary manifolds to jet bundles, which are endowed with their own, restrictive geometric structures such as the Cartan connection $\nabla_{\mathcal{C}}$ and which harbour systems of PDE. We prove now that the geometry of Lie algebra-valued connection $\mathfrak{g}$-forms α satisfying zero-curvature equation (6.3) gives rise to the geometry of solutions $\widehat{S}$ for the classical master-equation

$$\mathcal{E}_{\text{CME}} = \bigl\{\boldsymbol{i}\hbar\,\Delta\widehat{S}\big|_{\hbar=0} = \tfrac{1}{2}[\![\widehat{S}, \widehat{S}]\!]\bigr\}, \tag{6.1}$$

see Theorem 9 on p. 94 below. It is readily seen that realization (6.1) of the gauge-invariant setup is the classical limit of the full quantum picture as $\hbar \to 0$; the objective of quantization $\widehat{S} \longmapsto S^\hbar$ is a solution of the quantum master-equation

$$\mathcal{E}_{\text{QME}} = \bigl\{\boldsymbol{i}\hbar\,\Delta S^\hbar = \tfrac{1}{2}[\![S^\hbar, S^\hbar]\!]\bigr\} \tag{6.2}$$

for the true action functional $S^\hbar$ at $\hbar \neq 0$. Its construction involves quantum, noncommutative objects such as the deformations $\mathfrak{g}_\hbar$ of Lie algebras together with deformations of their duals (cf. [31]). (In fact, we express the notion of non-Abelian variational Lie algebroids in terms of the homological evolutionary vector field $\widehat{Q}$ and classical master-equation (6.1) viewing this construction as an intermediate step towards quantization.) A transition from the semiclassical to quantum picture results in $\mathfrak{g}_\hbar$-valued connections, quantum gauge groups, quantum vector spaces for values of the wave functions in auxiliary linear problems (6.4), and quantum extensions of physical fields.[1]

[1]Lie algebra-valued connection one-forms are the main objects in classical gauge field theories. Such

Remark 12. The geometry which we analyse in this chapter is produced and arranged by using the pull-backs $f^*(\varrho)$ of fibre bundles ϱ under some mappings f. Typically, the fibres of ϱ are Lie algebra-valued horizontal differential forms coming from $\Lambda^*(M^n)$, or similar objects[2]; in turn, the mappings f are projections to the base M^n of some infinite jet bundles. We employ the standard notion of *horizontal infinite jet bundles* such as $\overline{J^\infty_\xi}(\chi)$ or $\overline{J^\infty_\chi}(\xi)$ over infinite jet bundles $J^\infty(\xi)$ and $J^\infty(\chi)$, respectively; these spaces are present in Fig. 6.1 on p. 88 and they occur in (the proof of) Theorems 8 and 9 below. A proof of the convenient isomorphism $\overline{J^\infty_\xi}(\chi) \cong J^\infty(\xi \times_{M^n} \chi) = J^\infty(\xi) \times_{M^n} J^\infty(\chi)$ is written in [71], see also references therein. However, we recall further that, strictly speaking, the entire picture – with fibres which are inhabited by form-valued parity-even or parity-odd (duals of the) Lie algebra $\mathfrak{g}$ – itself is the image of a pull-back under the projection $\pi_\infty\colon J^\infty(\pi) \to M^n$ in the infinite jet bundle over the bundle π of physical fields. In other words, *sections* of those induced bundles are elements of Lie algebra etc., but all coefficients are differential functions in configurations of physical fields (which is obvious, e. g., from (6.3) in Definition 2 on the next page). Fortunately, it is the composite geometry of a fibre but not its location over the composite-structure base manifold which plays the main rôle in proofs of Theorems 8 and 9.

It is clear now that an attempt to indicate not only the bundles ξ or χ, $\Pi\chi^*$, $\Pi\xi$, and ξ^* which determine the intrinsic properties of objects but also to display the bundles that generate the pull-backs would make all proofs sound like the well-known poem about the house that Jack built.

Therefore, we *denote* the objects such as p_i or α and their mappings (see p. 91 or p. 95) *as if* they were just sections, $p_i \in \Gamma(\xi)$ and $\alpha \in \Gamma(\chi)$, of the bundles ξ and χ over the base M^n.

The Maurer–Cartan equation

Let us recall the definition of zero-curvature representation [94]. A horizontal one-form $\alpha \in \mathfrak{g} \otimes \Lambda^{0,1}(\mathcal{E}^\infty)$ is called a $\mathfrak{g}$-valued zero-curvature representation for $\mathcal{E}$ if α satisfies the Maurer–Cartan equation

$$\mathcal{E}_{\mathrm{MC}} = \left\{ \bar{\mathrm{d}}_h \alpha - \tfrac{1}{2}[\alpha, \alpha] \doteq 0 \right\} \tag{6.3}$$

by virtue of equation $\mathcal{E}$ and its differential consequences.

physical models are called *Abelian* –e.g., Maxwell's electrodynamics– or *non-Abelian* –here, consider the Yang–Mills theories with structure Lie groups $SU(2)$ or $SU(3)$– according to the commutation table for the underlying Lie algebra. This is why we say that variational Lie algebroids are (*non-*)*Abelian*— referring to the Lie algebra-valued connection one-forms α in the geometry of gauge-invariant zero-curvature representations for PDE.

[2] Let us specify at once that the geometries of prototype fibres in the bundles under study are described by $\mathfrak{g}$-, $\mathfrak{g}^*$-, $\Pi\mathfrak{g}$-, or $\Pi\mathfrak{g}^*$-valued (-1)-, zero-, one-, two-, and three-forms; the degree -1 corresponds to the module $D_1(M^n)$ of vector fields.

Given a zero-curvature representation $\alpha = A_i\,\mathrm{d}x^i$, the Maurer–Cartan equation $\mathcal{E}_{\mathrm{MC}}$ can be interpreted as the compatibility condition for the linear system

$$\Psi_{x^i} = A_i\Psi, \tag{6.4}$$

where $A_i \in \mathfrak{g} \otimes C^\infty(\mathcal{E}^\infty)$ and Ψ is the wave function, that is, Ψ is a (local) section of the principal fibre bundle $P(\mathcal{E}^\infty, G)$ with action of the gauge Lie group G on fibres; the Lie algebra of G is $\mathfrak{g}$. Then the system of equations

$$D_{x^i}A_j - D_{x^j}A_i + [A_i, A_j] = 0, \qquad 1 \leqslant i < j \leqslant n,$$

is equivalent to Maurer–Cartan's equation (6.3).

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G and α be a $\mathfrak{g}$-valued zero-curvature representation for a given PDE system $\mathcal{E}$. A gauge transformation $\Psi \mapsto g\Psi$ of the wave function by an element $g \in C^\infty(\mathcal{E}^\infty, G)$ induces the change

$$\alpha \mapsto \alpha^g = g \cdot \alpha \cdot g^{-1} + \bar{\mathrm{d}}_h g \cdot g^{-1}.$$

The zero-curvature representation α^g is called *gauge equivalent* to the initially given α; the G-valued function g on $\mathcal{E}^\infty$ determines the *gauge transformation* of α. For convenience, we make no distinction between the gauge transformations $\alpha \mapsto \alpha^g$ and G-valued functions g which generate them.

It is readily seen that a composition of two gauge transformations, by using g_1 first and then by g_2, itself is a gauge transformation generated by the G-valued function $g_2 \circ g_1$. Indeed, we have that

$$\begin{aligned}
(\alpha^{g_1})^{g_2} = (\bar{\mathrm{d}}_h g_1 \cdot g_1^{-1} + g_1 \cdot \alpha \cdot g_1^{-1})^{g_2} &= \bar{\mathrm{d}}_h g_2 \cdot g_2^{-1} + g_2 \cdot (\bar{\mathrm{d}}_h g_1 \cdot g_1^{-1} + g_1 \cdot \alpha \cdot g_1^{-1}) \cdot g_2^{-1} \\
&= (\bar{\mathrm{d}}_h g_2 \cdot g_1 + g_2 \cdot \bar{\mathrm{d}}_h g_1) \cdot g_1^{-1} \cdot g_2^{-1} + g_2 \cdot g_1 \cdot \alpha \cdot g_1^{-1} \cdot g_2^{-1} \\
&= \bar{\mathrm{d}}_h(g_2 \cdot g_1) \cdot (g_2 \cdot g_1)^{-1} + (g_2 \cdot g_1) \cdot \alpha \cdot (g_2 \cdot g_1)^{-1}.
\end{aligned}$$

We now consider *infinitesimal* gauge transformations generated by elements of the Lie group G which are close to its unit element $\mathbf{1}$. Suppose that $g_1 = \exp(\lambda p_1) = \mathbf{1} + \lambda p_1 + \frac{1}{2}\lambda^2 p_1^2 + o(\lambda^2)$ and $g_2 = \exp(\mu p_2) = \mathbf{1} + \mu p_2 + \frac{1}{2}\mu^2 p_2^2 + o(\mu^2)$ for some $p_1, p_2 \in \mathfrak{g}$ and $\mu, \lambda \in \mathbb{R}$. The following lemma, an elementary proof of which refers to the definition of Lie algebra, is the key to a construction of the anchors in non-Abelian variational Lie algebroids.

Lemma 3. Let α be a $\mathfrak{g}$-valued zero-curvature representation for a system $\mathcal{E}$. Then the commutant $g_1 \circ g_2 \circ g_1^{-1} \circ g_2^{-1}$ of infinitesimal gauge transformations g_1 and g_2 is an infinitesimal gauge transformation again.

Proof. By definition, put $g = g_1 \circ g_2 \circ g_1^{-1} \circ g_2^{-1}$. Taking into account that $g_1^{-1} = \mathbf{1} - \lambda p_1 + \frac{1}{2}\lambda^2 p_1^2 + o(\lambda^2)$ and $g_2^{-1} = \mathbf{1} - \mu p_2 + \frac{1}{2}\mu^2 p_2^2 + o(\mu^2)$, we obtain that

$$g = g_1 g_2 g_1^{-1} g_2^{-2} = \mathbf{1} + \lambda\mu \cdot (p_1 p_2 - p_2 p_1) + o(\lambda^2 + \mu^2).$$

We finally recall that $[p_1, p_2] \in \mathfrak{g}$, whence follows the assertion. $\square$

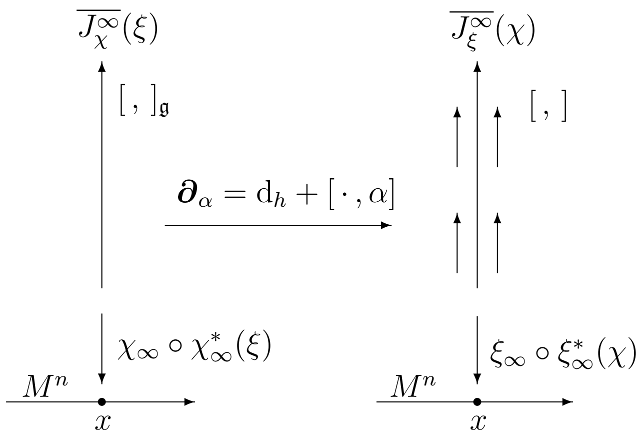

Figure 6.1: Non-Abelian variational Lie algebroid.

An infinitesimal gauge transformation $g = \mathbf{1} + \lambda p + o(\lambda)$ acts on a given $\mathfrak{g}$-valued zero-curvature representation α for an equation $\mathcal{E}^\infty$ by the formula

$$\begin{aligned}\alpha^g &= \bar{\mathrm{d}}_h(\mathbf{1} + \lambda p + o(\lambda)) \cdot (\mathbf{1} - \lambda p + o(\lambda)) + (\mathbf{1} + \lambda p + o(\lambda)) \cdot \alpha \cdot (\mathbf{1} - \lambda p + o(\lambda)) \\ &= \lambda \bar{\mathrm{d}}_h p + \alpha + \lambda(p\alpha - \alpha p) + o(\lambda) = \alpha + \lambda(\bar{\mathrm{d}}_h p + [p, \alpha]) + o(\lambda).\end{aligned}$$

From the coefficient of λ we obtain the operator $\bar{\boldsymbol{\partial}}_\alpha = \bar{\mathrm{d}}_h + [\cdot, \alpha]$. Lemma 3 implies that the image of this operator is closed under commutation in $\mathfrak{g}$, that is, $[\operatorname{im} \bar{\boldsymbol{\partial}}_\alpha, \operatorname{im} \bar{\boldsymbol{\partial}}_\alpha] \subseteq \operatorname{im} \bar{\boldsymbol{\partial}}_\alpha$. Such operators and their properties were studied in [70, 69]. We now claim that the operator $\bar{\boldsymbol{\partial}}_\alpha$ yields the anchor in a non-Abelian variational Lie algebroid, see Fig. 6.1; this construction is elementary (see Remark 12 on p. 86). Namely, the non-Abelian Lie algebroid $(\pi^*_\infty \circ \chi^*_\infty(\xi), \boldsymbol{\partial}_\alpha, [\,,\,]_{\mathfrak{g}})$ consists of

- the pull-back of the bundle ξ for $\mathfrak{g}$-valued gauge parameters p; the pull-back is obtained by using the bundle χ for $\mathfrak{g}$-forms α and (again by using the infinite jet bundle π_∞ over) the bundle π of physical fields,

- the (restriction $\bar{\boldsymbol{\partial}}_\alpha$ to $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ of the) anchor $\boldsymbol{\partial}_\alpha$ that generates infinitesimal gauge transformations $\dot{\alpha} = \boldsymbol{\partial}_\alpha(p)$ in the bundle χ of $\mathfrak{g}$-valued connection one-forms, and

- the Lie algebra structure $[\,,\,]_{\mathfrak{g}}$ on the anchor's domain of definition.

We refer to [68] for more detail and for discussion on that object's structural complexity.

Noether identities for the Maurer–Cartan equation

In the meantime, let us discuss Noether identities [12, 62, 106] for Maurer–Cartan equation (6.3). Depending on the dimension n of the base manifold M^n, we consider the cases

$n = 2$, $n = 3$, and $n > 3$. We suppose that the Lie algebra $\mathfrak{g}$ is equipped[3] with a nondegenerate ad-invariant metric t_{ij}. The paring $\langle\,,\,\rangle$ is defined for elements of $\mathfrak{g} \otimes \Lambda(M^n)$ as follows,

$$\langle A\mu, B\nu\rangle = \langle A, B\rangle\, \mu \wedge \nu,$$

where the coupling $\langle A, B\rangle$ is given by the metric t_{ij} for $\mathfrak{g}$. From the ad-invariance $\langle [A, B], C\rangle = \langle A, [B, C]\rangle$ of the metric t_{ij} we deduce that

$$\begin{aligned}\langle [A\mu, B\nu], C\rho\rangle = \langle [A, B]\, \mu \wedge \nu, C\rho\rangle = \langle [A, B], C\rangle\, \mu \wedge \nu \wedge \rho = \langle A, [B, C]\rangle\, \mu \wedge \nu \wedge \rho \\ = \langle A\mu, [B, C]\, \nu \wedge \rho\rangle = \langle A\mu, [B\nu, C\rho]\rangle.\end{aligned}$$

Let us denote by $\mathcal{F} = -\mathrm{d}_h\alpha + \frac{1}{2}[\alpha, \alpha]$ the left-hand side of Maurer–Cartan equation (6.3). We recall from that $\dot\alpha = \boldsymbol{\partial}_\alpha(p)$ is a gauge symmetry of Maurer–Cartan equation (6.3). Moreover, for all $n > 1$ the operator $\boldsymbol{\partial}_\alpha^\dagger$ produces a Noether identity for (6.3), which is readily seen from the following statement.

Proposition 8. The left-hand sides $\mathcal{F} = -\mathrm{d}_h\alpha + \frac{1}{2}[\alpha, \alpha]$ of Maurer–Cartan's equation satisfy the Noether identity (or *Bianchi identity* for the curvature two-form)

$$\boldsymbol{\partial}_\alpha^\dagger(\mathcal{F}) = -\mathrm{d}_h\mathcal{F} - [\mathcal{F}, \alpha] \equiv 0. \tag{6.5}$$

Proof. Applying the operator $\boldsymbol{\partial}_\alpha^\dagger$ to the left-hand sides of Maurer–Cartan's equation, we obtain

$$\begin{aligned}\boldsymbol{\partial}_\alpha^\dagger(\mathcal{F}) = \boldsymbol{\partial}_\alpha^\dagger\bigl(-\mathrm{d}_h\alpha + \tfrac{1}{2}[\alpha, \alpha]\bigr) = (-\mathrm{d}_h - [\cdot, \alpha])\bigl(-\mathrm{d}_h\alpha + \tfrac{1}{2}[\alpha, \alpha]\bigr) = \\ = (\mathrm{d}_h \circ \mathrm{d}_h)\alpha - \tfrac{1}{2}\mathrm{d}_h\bigl([\alpha, \alpha]\bigr) + [\mathrm{d}_h\alpha, \alpha] - \tfrac{1}{2}[\alpha, [\alpha, \alpha]] = \\ = -[\mathrm{d}_h\alpha, \alpha] + [\mathrm{d}_h\alpha, \alpha] - \tfrac{1}{2}[\alpha, [\alpha, \alpha]] = 0.\end{aligned}$$

The third term in the last line is zero due to the Jacobi identity, whereas the first two cancel out. ☐

Let $n = 2$. The Maurer–Cartan equation's left-hand sides $\mathcal{F}$ are top-degree forms, hence every operator which increases the form degree vanishes at $\mathcal{F}$.

Consider the case $n = 3$; we recall that Maurer–Cartan equation (6.3) is Euler–Lagrange in this setup (cf. [2, 3, 124]).

Proposition 9. If the base manifold M^3 is 3-dimensional, then Maurer–Cartan's equation is Euler–Lagrange with respect to the action functional

$$S_{\mathrm{MC}} = \int \boldsymbol{\mathcal{L}} = \int \left\{-\tfrac{1}{2}\langle \alpha, \mathrm{d}_h\alpha\rangle + \tfrac{1}{6}\langle \alpha, [\alpha, \alpha]\rangle\right\}. \tag{6.6}$$

Note that its Lagrangian density $\boldsymbol{\mathcal{L}}$ is a well-defined top-degree form on the base threefold M^3.

[3]Notice that the Lie algebra $\mathfrak{g}$ is canonically identified with its dual $\mathfrak{g}^*$ via nondegenerate metric t_{ij}.

Proof. Let us construct the Euler–Lagrange equation:

$$\delta \int \left\{ -\tfrac{1}{2}\langle \alpha, \mathrm{d}_h\alpha\rangle + \tfrac{1}{6}\langle \alpha, [\alpha,\alpha]\rangle \right\} = \langle \delta\alpha, -\mathrm{d}_h\alpha\rangle + \tfrac{1}{6}(\langle \delta\alpha, [\alpha,\alpha]\rangle + \langle \alpha, [\delta\alpha,\alpha]\rangle + \langle \alpha, [\alpha, \delta\alpha]\rangle$$
$$= \langle \delta\alpha, -\mathrm{d}_h\alpha + \tfrac{1}{2}[\alpha,\alpha]\rangle.$$

This proves our claim. □

Proposition 10. For each $p \in \mathfrak{g}\otimes\Lambda^0(M^3)$, the evolutionary vector field $\vec{\partial}^{(\alpha)}_{A(p)}$ with generating section $A(p) = \boldsymbol{\partial}_\alpha(p) = \mathrm{d}_h p + [p,\alpha]$ is a Noether symmetry of the action S_{MC},[4]

$$\vec{\partial}^{(\alpha)}_{A(p)}(S_{\mathrm{MC}}) \cong 0 \in \overline{H}^n(\chi).$$

The operator $A = \boldsymbol{\partial}_\alpha = \mathrm{d}_h + [\cdot,\alpha]$ determines linear Noether's identity (6.5),

$$\Phi(x,\alpha,\mathcal{F}) = A^\dagger(\mathcal{F}) \equiv 0,$$

for left-hand sides of the system of Maurer–Cartan's equations (6.3).

Proof. We have

$$\vec{\partial}^{(\alpha)}_{A(p)} S_{\mathrm{MC}} \cong \langle A(p), \tfrac{\delta}{\delta\alpha} S_{\mathrm{MC}}\rangle \cong \left\langle \left(\ell_\Phi^{(\mathcal{F})}\right)^\dagger(p), \mathcal{F}\right\rangle \cong \langle p, \ell_\Phi^{(\mathcal{F})}(\mathcal{F})\rangle = \langle p, \Phi(\mathcal{F})\rangle = \langle p, A^\dagger(\mathcal{F})\rangle.$$

In Proposition 8 we prove that $A^\dagger(\mathcal{F}) \equiv 0$. So for all p we have that $\langle p, A^\dagger(\mathcal{F})\rangle \cong 0$, which concludes the proof. □

Finally, we let $n > 3$. In this case of higher dimension, the Lagrangian $\mathcal{L} = \langle \alpha, \frac{1}{6}[\alpha,\alpha] - \frac{1}{2}\mathrm{d}_h\alpha\rangle \in \Lambda^3(M^n)$ does not belong to the space of top-degree forms and Proposition 9 does not hold. However, Noether's identity $\boldsymbol{\partial}_\alpha^\dagger(\mathcal{F}) \equiv 0$ still holds if $n > 3$ according to Proposition 8.

6.1 Non-Abelian variational Lie algebroids

Let $\vec{e}_1, \ldots, \vec{e}_d$ be a basis in the Lie algebra $\mathfrak{g}$. Every $\mathfrak{g}$-valued zero-curvature representation for a given PDE system $\mathcal{E}^\infty$ is then $\alpha = \alpha_i^k \vec{e}_k\,\mathrm{d}x^i$ for some coefficient functions $\alpha_i^k \in C^\infty(\mathcal{E}^\infty)$. Construct the vector bundle $\chi\colon \Lambda^1(M^n)\otimes\mathfrak{g} \to M^n$ and the trivial bundle $\xi\colon M^n\times\mathfrak{g}\to M^n$ with the Lie algebra $\mathfrak{g}$ taken for fibre. Next, introduce the superbundle $\Pi\xi\colon M^n\times\Pi\mathfrak{g}\to M^n$ the total space of which is the same as that of ξ but such that the parity of fibre coordinates is reversed[5]). Finally, consider the Whitney sum $J^\infty(\chi)\times_{M^n} J^\infty(\Pi\xi)$ of infinite jet bundles over the parity-even vector bundle χ and parity-odd $\Pi\xi$.

[4]Here $\cong$ denotes the equality up to integration by parts and we assume the absence of boundary terms.

[5]The odd neighbour $\Pi\mathfrak{g}$ of the Lie algebra is introduced in order to handle poly-linear, totally skew-symmetric maps of elements of $\mathfrak{g}$ so that the parity-odd space $\Pi\mathfrak{g}$ carries the information about the Lie algebra's structure constants c_{ij}^k still not itself becoming a Lie superalgebra.

With the geometry of every $\mathfrak{g}$-valued zero-curvature representation we associate a non-Abelian variational Lie algebroid [70]. Its realization by a homological evolutionary vector field is the differential in the arising gauge cohomology theory (cf. [120] and [3, 49, 70, 74, 94]).

Theorem 8 ([68])**.** The parity-odd evolutionary vector field which encodes the non-Abelian variational Lie algebroid structure on the infinite jet superbundle $J^\infty(\chi\times_{M^n}\Pi\xi)\cong J^\infty(\chi)\times_{M^n}J^\infty(\Pi\xi)$ is

$$Q=\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b}+\tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]},\qquad [Q,Q]=0\quad\Longleftrightarrow\quad Q^2=0, \tag{6.7}$$

where for each choice of respective indexes,

- α^k_μ is a parity-even coordinate along fibres in the bundle χ of $\mathfrak{g}$-valued one-forms,
- b^k is a parity-odd fibre coordinate in the bundle $\Pi\xi$,
- c^k_{ij} is a structure constant in the Lie algebra $\mathfrak{g}$ so that $[b^i,b^j]^k=b^ic^k_{ij}b^j$ and $[b^i,\alpha^j]^k=b^ic^k_{ij}\alpha^j$,
- d_h is the horizontal differential on the Whitney sum of infinite jet bundles,
- the operator $\boldsymbol{\partial}_\alpha=\mathrm{d}_h+[\cdot,\alpha]\colon\overline{J^\infty_\chi}(\Pi\xi)\cong J^\infty(\chi\times_{M^n}\Pi\xi)\to\overline{J^\infty_{\Pi\xi}}(\chi)\cong J^\infty(\chi\times_{M^n}\Pi\xi)$ is the anchor.

Proof. The anticommutator $[Q,Q]=2Q^2$ of the parity-odd vector field Q with itself is again an evolutionary vector field. Therefore it suffices to prove that the coefficients of $\vec{\partial}/\partial\alpha$ and $\vec{\partial}/\partial b$ are equal to zero in the vector field

$$Q^2=\left(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b}+\tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]}\right)\left(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b}+\tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]}\right).$$

We have $[b,b]^k=b^ic^k_{ij}b^j$ by definition. Hence it is readily seen that $(\frac{1}{2}\vec{\partial}^{(b)}_{b^ic^k_{ij}b^j})^2=0$ because $\mathfrak{g}$ is a Lie algebra [122] so that the Jacobi identity is satisfied by the structure constants. Since the bracket $[b,b]$ does not depend on α, we deduce that $(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b})(\frac{1}{2}\vec{\partial}^{(b)}_{[b,b]})=0$. Therefore,

$$\begin{aligned}Q^2=\left(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b}+\tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]}\right)\left(\vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h b}\right)&=-\vec{\partial}^{(\alpha)}_{[b,[b,\alpha]+\mathrm{d}_h b]}+\tfrac{1}{2}\vec{\partial}^{(\alpha)}_{[[b,b],\alpha]+\mathrm{d}_h([b,b])}\\&=\vec{\partial}^{(\alpha)}_{-[b,[b,\alpha]+\mathrm{d}_h b]+\frac{1}{2}[[b,b],\alpha]+\frac{1}{2}\mathrm{d}_h([b,b])}.\end{aligned}$$

Now consider the expression $-[b,[b,\alpha]+\mathrm{d}_h b]+\frac{1}{2}[[b,b],\alpha]+\frac{1}{2}\mathrm{d}_h([b,b])$, viewing it as a bilinear skew-symmetric map $\Gamma(\xi)\times\Gamma(\xi)\to\Gamma(\chi)$. First, we claim that the value $\big(\frac{1}{2}[[b,b],\alpha]-$

$[b,[b,\alpha]]\big)(p_1,p_2)$ at any two sections p_1, $p_2 \in \Gamma(\xi)$ vanishes identically. Indeed, by taking an alternating sum over the permutation group of two elements we have that

$$\begin{aligned}\tfrac{1}{2}[[p_1,p_2],\alpha]-\tfrac{1}{2}[[p_2,p_1],\alpha]-[p_1,[p_2,\alpha]]+[p_2,[p_1,\alpha]] &= [[p_1,p_2],\alpha]-[p_1,[p_2,\alpha]]-[p_2,[\alpha,p_1]]\\ &= -[\alpha,[p_1,p_2]]-[p_1,[p_2,\alpha]]-[p_2,[\alpha,p_1]=0.\end{aligned}$$

At the same time, the value of bi-linear skew-symmetric mapping $\frac{1}{2}\mathrm{d}_h([b,b])-[b,\mathrm{d}_h b]$ at sections p_1 and p_2 also vanishes,

$$\tfrac{1}{2}\mathrm{d}_h([p_1,p_2])-\tfrac{1}{2}\mathrm{d}_h([p_2,p_1])-[p_1,\mathrm{d}_h p_2]+[p_2,\mathrm{d}_h p_1]=\mathrm{d}_h([p_1,p_2])-[p_1,\mathrm{d}_h p_2]-[\mathrm{d}_h p_1,p_2]=0.$$

We conclude that

$$Q^2\Big|_{(p_1,p_2)}=\vec{\partial}^{(\alpha)}_{\{-[b,[b,\alpha]+\mathrm{d}_h b]+\frac{1}{2}[[b,b],\alpha]+\frac{1}{2}\mathrm{d}_h([b,b])\}(p_1,p_2)}=\vec{\partial}^{(\alpha)}_0=0,$$

which proves the theorem. □

Finally, let us derive a reparametrization formula for the homological vector field Q in the course of gauge transformations of zero-curvature representations. We begin with some trivial facts [16, 35].

Lemma 4. Let α be a $\mathfrak{g}$-valued zero-curvature representation for a PDE system. Consider two infinitesimal gauge transformations given by $g_1=\mathbf{1}+\varepsilon p_1+o(\varepsilon)$ and $g_2=\mathbf{1}+\varepsilon p_2+o(\varepsilon)$. Let $g\in C^\infty(\mathcal{E}^\infty,G)$ also determine a gauge transformation. Then the following diagram is commutative,

$$\begin{array}{ccc}\alpha^g & \xrightarrow{g_2} & \beta\\ \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle g}\\ \alpha & \xrightarrow{g_1} & \alpha^{g_1},\end{array}$$

if the relation $p_2=g\cdot p_1\cdot g^{-1}$ is valid.

Proof. By the lemma's assumption we have that $(\alpha^{g_1})^g=(\alpha^g)^{g_2}$. Hence we deduce that

$$g\cdot(\mathbf{1}+\varepsilon p_1)=(\mathbf{1}+\varepsilon p_2)\cdot g \qquad\Longleftrightarrow\qquad g\cdot p_1=p_2\cdot g,$$

which yields the transformation rule $p_2=g\cdot p_1\cdot g^{-1}$ for the $\mathfrak{g}$-valued function p_1 on $\mathcal{E}^\infty$ in the course of gauge transformation $g\colon\alpha\mapsto\alpha^g$. □

Using the above lemma we describe the behaviour of homological vector field Q in the non-Abelian variational setup of Theorem 8.

Corollary 1. Under a coordinate change

$$\alpha\mapsto\alpha'=g\cdot\alpha\cdot g^{-1}+d_h g\cdot g^{-1},\qquad b\mapsto b'=g\cdot b\cdot g^{-1},$$

where $g\in C^\infty(M^n,G)$, the variational Lie algebroid's differential Q is transformed accordingly:

$$Q\longmapsto Q'=\vec{\partial}^{(\alpha')}_{[b',\alpha']+\mathrm{d}_h b'}+\tfrac{1}{2}\vec{\partial}^{(b')}_{[b',b']}.$$

6.2 The master-functional for zero-curvature representations

The correspondence between zero-curvature representations, i.e., classes of gauge-equivalent solutions α to the Maurer–Cartan equation, and non-Abelian variational Lie algebroids goes in parallel with the BRST-technique, in the frames of which ghost variables appear and gauge algebroids arise (see [6, 61]). Let us therefore extend the BRST-setup of fields α and ghosts b to the full BV-zoo of (anti)fields α and α^* and (anti)ghosts b and b^* (cf. [7, 8, 9, 46, 119]). We note that a finite-dimensional 'forefather' of what follows is discussed in detail in [3], which is devoted to Q- and QP-structures on (super)manifolds. Those concepts are standard; our message is that not only the approach of [3] to QP-structures on G-manifolds X and $\Pi T^*\bigl(X \times \Pi TG/G\bigr) \simeq \Pi T^*X \times \mathfrak{g}^* \times \Pi\mathfrak{g}$ remains applicable in the variational setup of jet bundles (i.e., whenever integrations by parts are allowed, whence many Leibniz rule structures are lost), but even the explicit formulas for the BRST-field Q and the action functional $\widehat{S}$ for the extended field $\widehat{Q}$ are valid literally. In fact, we recover the *third* and *fourth* equivalent formulations of the definition for a variational Lie algebroid (cf. [3, 120] or a review [75]).

Let us recall from section 6.1 that α is a tuple of even-parity fibre coordinates in the bundle $\chi\colon \Lambda^1(M^n) \otimes \mathfrak{g} \to M^n$ and b are the odd-parity coordinates along fibres in the trivial vector bundle $\Pi\xi\colon M^n \times \Pi\mathfrak{g} \to M^n$. We now let all the four *neighbours* of the Lie algebra $\mathfrak{g}$ appear on the stage: they are $\mathfrak{g}$ (in χ), $\mathfrak{g}^*$, $\Pi\mathfrak{g}$ (in $\Pi\xi$), and $\Pi\mathfrak{g}^*$ (see [122] and reference therein). Let us consider the bundle $\Pi\chi^*\colon \mathrm{D}_1(M^n) \otimes \Pi\mathfrak{g}^* \to M^n$ whose fibres are dual to those in χ and also have the parity reversed.[6] We denote by α^* the collection of odd fibre coordinates in $\Pi\chi^*$.

Remark 13. In what follows we do not write the (indexes for) bases of vectors in the fibres of $\mathrm{D}_1(M^n)$ or of covectors in $\Lambda^1(M^n)$; to make the notation short, their couplings are implicit. Nevertheless, a summation over such "invisible" indexes in $\partial/\partial x^\mu$ and $\mathrm{d}x^\nu$ is present in all formulas containing the couplings of α and α^*. We also note that $(\alpha^*)\,\overleftarrow{\mathrm{d}_h}$ is a very interesting object because α^* parametrizes fibres in $\mathrm{D}_1(M^n) \otimes \Pi\mathfrak{g}^*$; the horizontal differential d_h produces the forms $\mathrm{d}x^i$ which are initially not coupled with their duals from $\mathrm{D}_1(M^n)$. (However, such objects cancel out in the identity $\widehat{Q}^2 = 0$, see (6.11) on p. 95.)

Secondly, we consider the even-parity dual $\xi^*\colon M^n \times \mathfrak{g}^* \to M^n$ of the odd bundle $\Pi\xi$; let us denote by b^* the coordinates along $\mathfrak{g}^*$ in the fibres of ξ^*.

Finally, we fix the ordering

$$\delta\alpha \wedge \delta\alpha^* + \delta b^* \wedge \delta b \tag{6.8}$$

[6] In terms of [3], the Whitney sum $J^\infty(\chi) \times_{M^n} J^\infty(\Pi\chi^*)$ plays the rôle of ΠT^*X for a G-manifold X; here $\mathfrak{g}$ is the Lie algebra of a Lie group G so that $\Pi\mathfrak{g} \simeq \Pi TG/G$.

of the canonically conjugate pairs of coordinates. By picking a volume form $\mathrm{dvol}(M^n)$ on the base M^n we then construct the odd Poisson bracket (variational Schouten bracket $[\![\,,\,]\!]$) on the senior d_h-cohomology (or *horizontal* cohomology) space $\overline{H}^n(\chi \times_{M^n} \Pi\chi^* \times_{M^n} \Pi\xi \times_{M^n} \xi^*)$; we refer to [63, 64] for a geometric theory of variations.

Theorem 9 ([68])**.** The structure of non-Abelian variational Lie algebroid from Theorem 8 is encoded on the Whitney sum $J^\infty(\chi \times_{M^n} \Pi\chi^* \times_{M^n} \Pi\xi \times_{M^n} \xi^*)$ of infinite jet (super)bundles by the action functional

$$\widehat{S} = \int \mathrm{dvol}(M^n)\,\bigl\{\langle \alpha^*, [b,\alpha] + \mathrm{d}_h(b)\rangle + \tfrac{1}{2}\langle b^*, [b,b]\rangle\bigr\} \in \overline{H}^n(\chi \times_{M^n} \Pi\chi^* \times_{M^n} \Pi\xi \times_{M^n} \xi^*)$$

which satisfies the classical master-equation

$$[\![\widehat{S}, \widehat{S}]\!] = 0.$$

The functional $\widehat{S}$ is the Hamiltonian of odd-parity evolutionary vector field $\widehat{Q}$ which is defined on $J^\infty(\chi) \times_{M^n} J^\infty(\Pi\chi^*) \times_{M^n} J^\infty(\Pi\xi) \times_{M^n} J^\infty(\xi^*)$ by the equality

$$\widehat{Q}(\mathcal{H}) \cong [\![\widehat{S}, \mathcal{H}]\!] \tag{6.9}$$

for any $\mathcal{H} \in \overline{H}^n(\chi \times_{M^n} \Pi\chi^* \times_{M^n} \Pi\xi \times_{M^n} \xi^*)$. The odd-parity field is[7]

$$\widehat{Q} = \vec{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h(b)} + \vec{\partial}^{(\alpha^*)}_{(\alpha^*)\overleftarrow{\mathrm{ad}}{}^*_b} + \tfrac{1}{2}\vec{\partial}^{(b)}_{[b,b]} + \vec{\partial}^{(b^*)}_{-\,\mathrm{ad}^*_\alpha(\alpha^*)+(\alpha^*)\overleftarrow{\mathrm{d}_h}+\mathrm{ad}^*_b(b^*)}, \tag{6.10}$$

where $\langle(\alpha^*)\,\overleftarrow{\mathrm{ad}}{}^*_b, \alpha\rangle \stackrel{\text{def}}{=} \langle\alpha^*, [b,\alpha]\rangle$ and $\langle \mathrm{ad}^*_b(b^*), p\rangle = \langle b^*, [b,p]\rangle$ for any $\alpha \in \Gamma(\chi)$ and $p \in \Gamma(\xi)$. This evolutionary vector field is homological,

$$\widehat{Q}^2 = 0.$$

Proof. In coordinates, the master-action $\widehat{S} = \int \widehat{\mathcal{L}}\ \mathrm{dvol}(M^n)$ is equal to

$$\widehat{S} = \int \mathrm{dvol}(M^n)\,\bigl\{\alpha^*_a(b^\mu c^a_{\mu\nu}\alpha^\nu + \mathrm{d}_h(b^a)) + \tfrac{1}{2} b^*_\mu b^\beta c^\mu_{\beta\gamma} b^\gamma\bigr\}\,;$$

here the summation over spatial degrees of freedom from the base M^n in implicit in the horizontal differential d_h and the respective contractions with α^*. By the Jacobi identity for the variational Schouten bracket $[\![\,,\,]\!]$ (see [64]), the classical master equation $[\![\widehat{S}, \widehat{S}]\!] = 0$

[7]The referee points out that the evolutionary vector field $\widehat{Q}$ is the jet-bundle upgrade of the *cotangent lift* of the field Q, which is revealed by the explicit formula for the Hamiltonian $\widehat{S}$. Let us recall that the cotangent lift of a vector field $\mathcal{Q} = \mathcal{Q}^i\,\partial/\partial q^i$ on a (super)manifold N^m is the Hamiltonian vector field on T^*N^m given by $\widehat{\mathcal{Q}} = \mathcal{Q}^i(q)\,\partial/\partial q^i - p_j \cdot \partial\mathcal{Q}^j(q)/\partial q^i\,\partial/\partial p_i$; its Hamiltonian is $\mathcal{S} = p_i\,\mathcal{Q}^i(q)$. An example of this classical construction is contained in the seminal paper [3].

is equivalent to the homological condition $\widehat{Q}^2 = 0$ for the odd-parity vector field defined by (6.9). The conventional choice of signs (6.8) yields a formula for this graded derivation,

$$\widehat{Q} = \vec{\partial}^{(\alpha)}_{-\vec{\delta}\widehat{\mathcal{L}}/\delta\alpha^*} + \vec{\partial}^{(\alpha^*)}_{\vec{\delta}\widehat{\mathcal{L}}/\delta\alpha} + \vec{\partial}^{(b)}_{\vec{\delta}\widehat{\mathcal{L}}/\delta b^*} + \vec{\partial}^{(b^*)}_{-\vec{\delta}\widehat{\mathcal{L}}/\delta b},$$

where the arrows over $\vec{\partial}$ and $\vec{\delta}$ indicate the direction along which the graded derivations act and graded variations are transported (that is, from left to right and rightmost, respectively). We explicitly obtain that[8]

$$\widehat{Q} = \vec{\partial}^{(\alpha^a)}_{b^\mu c^a_{\mu\nu}\alpha^\nu + \mathrm{d}_h(b^a)} + \vec{\partial}^{(\alpha^*_\nu)}_{\alpha^*_a b^\mu c^a_{\mu\nu}} + \vec{\partial}^{(b^\mu)}_{\frac{1}{2}b^\beta c^\mu_{\beta\gamma}b^\gamma} + \vec{\partial}^{(b^*_\mu)}_{\{-\alpha^*_a c^a_{\mu\nu}\alpha^\nu + (\alpha^*_\mu)\overleftarrow{\mathrm{d}_h} + b^*_a c^a_{\mu\nu}b^\nu\}}.$$

Actually, the proof of Theorem 8 contains the first half of a reasoning which shows why $\widehat{Q}^2 = 0$. (It is clear that the field $\widehat{Q}$ consists of (6.7) not depending on α^* and b^* and of the two new terms.) Again, the anticommutator $[\widehat{Q}, \widehat{Q}] = 2\widehat{Q}^2$ is an evolutionary vector field. We claim that the coefficients of $\vec{\partial}/\partial\alpha^*_\nu$ and $\vec{\partial}/\partial b^*_\mu$ in it are equal to zero.

Let us consider first the coefficient of $\vec{\partial}/\partial\alpha^*$ at the bottom of the evolutionary derivation $\vec{\partial}^{(\alpha^*)}_{\{\ldots\}}$ in $\widehat{Q}^2$; by contracting this coefficient with $\alpha = (\alpha^\nu)$ we obtain

$$\langle \alpha^*_a, b^\lambda c^a_{\lambda\mu} b^q c^\mu_{q\nu}\alpha^\nu - \tfrac{1}{2} b^\beta c^\mu_{\beta\gamma} b^\gamma c^a_{\mu\nu}\alpha^\nu \rangle.$$

It is readily seen that α^* is here coupled with the bi-linear skew-symmetric operator $\Gamma(\xi) \times \Gamma(\xi) \to \Gamma(\chi)$ for any fixed $\alpha \in \Gamma(\chi)$, and we show that this operator is zero on its domain of definition. Indeed, the comultiple $|\,\rangle$ of $\langle\alpha^*|$ is $[b, [b, \alpha]] - \frac{1}{2}[[b, b], \alpha]$ so that its value at any arguments $p_1, p_2 \in \Gamma(\xi)$ equals

$$[p_1, [p_2, \alpha]] - [p_2, [p_1, \alpha]] - [\tfrac{1}{2}[p_1, p_2] - \tfrac{1}{2}[p_2, p_1], \alpha] = 0$$

by the Jacobi identity.

Let us now consider the coefficient of $\vec{\partial}/\partial b^*_\mu$ in the vector field $\widehat{Q}^2$,

$$\begin{aligned} &- \left[\alpha^*_{\tilde{a}} b^{\tilde{\mu}} c^{\tilde{a}}_{\tilde{\mu}a}\right] c^a_{\mu\nu}\alpha^\nu + \alpha^*_a c^a_{\mu\nu}\left[b^{\tilde{\mu}} c^\nu_{\tilde{\mu}\tilde{\nu}}\alpha^{\tilde{\nu}} + \mathrm{d}_h(b^\nu)\right] + \left(\left[\alpha^*_{\tilde{a}} b^{\tilde{\mu}} c^{\tilde{a}}_{\tilde{\mu}\mu}\right]\right)\overleftarrow{\mathrm{d}_h} \\ &\qquad + \left[-\alpha^*_{\tilde{a}} c^{\tilde{a}}_{a\tilde{\nu}}\alpha^{\tilde{\nu}} + (\alpha^*_a)\overleftarrow{\mathrm{d}_h} + b^*_{\tilde{a}} c^{\tilde{a}}_{a\tilde{\nu}} b^{\tilde{\nu}}\right] c^a_{\mu\nu} b^\nu + b^*_a c^a_{\mu\nu}\cdot\left[\tfrac{1}{2} b^{\tilde{\beta}} c^\nu_{\tilde{\beta}\tilde{\gamma}} b^{\tilde{\gamma}}\right]; \end{aligned}$$

here we mark with a tilde sign those summation indexes which come from the first copy of $\widehat{Q}$ acting from the left on $\vec{\partial}^{(b^*_\mu)}_{\{\ldots\}}$ in $\widehat{Q}\circ\widehat{Q}$. Two pairs of cancellations occur in the terms which contain the horizontal differential d_h. First, let us consider the terms in which the differential acts on α^*. By contracting the index μ with an extra copy $b = (b^\mu)$, we obtain

$$(\alpha^*_a)\overleftarrow{\mathrm{d}_h}\, b^\lambda c^a_{\lambda\mu} b^\mu + (\alpha^*_a)\overleftarrow{\mathrm{d}_h}\, c^a_{\mu\lambda} b^\lambda b^\mu. \tag{6.11}$$

[8]Note that $\langle \alpha^*, \overrightarrow{\mathrm{d}_h}(b)\rangle \cong -\langle(\alpha^*)\overleftarrow{\mathrm{d}_h}, b\rangle$ in the course of integration by parts, whence the term $(\alpha^*_\mu)\overleftarrow{\mathrm{d}_h}$ that comes from $-\vec{\delta}\widehat{\mathcal{L}}/\delta b^\mu$ does stand with a plus sign in the velocity of b^*_μ.

Due to the skew-symmetry of structure constants c^k_{ij} in $\mathfrak{g}$, at any sections $p_1, p_2 \in \Gamma(\xi)$ we have that

$$(\alpha^*_a) \overleftarrow{\mathrm{d}_h} \cdot \left(p_1^\lambda c^a_{\lambda\mu} p_2^\mu - p_2^\lambda c^a_{\lambda\mu} p_1^\mu + c^a_{\mu\lambda} p_1^\lambda p_2^\mu - c^a_{\mu\lambda} p_2^\lambda p_1^\mu\right) = 0.$$

Likewise, a contraction with $b = (b^\mu)$ for the other pair of terms with d_h, now acting on b, yields

$$\alpha^*_a\, c^a_{\mu\lambda}\, \mathrm{d}_h(b^\lambda) b^\mu + \alpha^*_a\, \mathrm{d}_h(b^\lambda)\, c^a_{\lambda\mu} b^\mu. \tag{6.12}$$

At the moment of evaluation at p_1 and p_2, expression (6.12) cancels out due to the same mechanism as above.

The remaining part of the coefficient of $\vec{\partial}/\partial b^*_\mu$ in $\widehat{Q}^2$ is

$$\begin{multlined} -\alpha^*_z b^\lambda c^z_{\lambda a} c^a_{\mu\nu} \alpha^\nu + \alpha^*_z c^z_{\mu\nu} b^i c^\nu_{ij} \alpha^j - \alpha^*_z c^z_{a\nu} \alpha^\nu c^a_{\mu j} b^j \\ + b^*_\lambda c^\lambda_{a\gamma} b^\gamma c^a_{\mu j} b^j + b^*_\lambda c^\lambda_{\mu\gamma} \cdot \tfrac{1}{2} b^\beta c^\gamma_{\beta\delta} b^\delta. \end{multlined} \tag{6.13}$$

It is obvious that the mechanisms of vanishing are different for the first and second lines in (6.13) whenever each of the two is regarded as mapping which takes $b = (b^\mu)$ to a number from the field $\Bbbk$. Therefore, let us consider these two lines separately.

By contracting the upper line of (6.13) with $b = (b^\mu)$, we rewrite it as follows,

$$\langle -\alpha^*_z, b^\lambda c^z_{\lambda a} c^a_{\mu\nu} \alpha^\nu b^\mu - c^z_{\mu\nu} b^i c^\nu_{ij} \alpha^j b^\mu + c^z_{a\nu} \alpha^\nu c^a_{\mu j} b^j b^\mu \rangle.$$

Viewing the content of the co-multiple $|\rangle$ of $\langle -\alpha^*|$ as bi-linear skew-symmetric mapping $\Gamma(\xi) \times \Gamma(\xi) \to \Gamma(\chi)$, we conclude that its value at any pair of section $p_1, p_2 \in \Gamma(\xi)$ is

$$\begin{aligned} &[p_2, [p_1, \alpha]] - [p_1, [p_2, \alpha]] + [[p_1, p_2], \alpha] \\ -\, &[p_1, [p_2, \alpha]] + [p_2, [p_1, \alpha]] - [[p_2, p_1], \alpha] = 0 - 0 = 0, \end{aligned}$$

because each line itself amounts to the Jacobi identity.

At the same time, the contraction of lower line in (6.13) with $b = (b^\mu)$ gives

$$\langle b^*_\lambda, c^\lambda_{a\gamma} b^\gamma c^a_{\mu j} b^j b^\mu + c^\lambda_{\mu\gamma} \cdot \tfrac{1}{2} b^\beta c^\gamma_{\beta\delta} b^\delta b^\mu \rangle.$$

The term $|\rangle$ near $\langle b^*|$ determines the tri-linear skew-symmetric mapping $\Gamma(\xi) \times \Gamma(\xi) \times \Gamma(\xi) \to \Gamma(\xi)$ whose value at any $p_1, p_2, p_3 \in \Gamma(\xi)$ is defined by the formula

$$\sum_{\sigma \in S_3} (-)^\sigma \Big\{ \big[[p_{\sigma(1)}, p_{\sigma(2)}], p_{\sigma(3)}\big] + \big[p_{\sigma(1)}, \tfrac{1}{2}[p_{\sigma(2)}, p_{\sigma(3)}]\big] \Big\}.$$

This amounts to four copies of the Jacobi identity (indeed, let us take separate sums over even and odd permutations). Consequently, the tri-linear operator at hand, hence the entire coefficient of $\vec{\partial}/\partial b^*$, is equal to zero so that $\widehat{Q}^2 = 0$. $\square$

Let us sum up the geometries we are dealing with. We started with a partial differential equation $\mathcal{E}$ for physical fields; it is possible that $\mathcal{E}$ itself was Euler–Lagrange[9] and it could be gauge-invariant with respect to some Lie group. We then recalled the notion of $\mathfrak{g}$-valued zero-curvature representations α for $\mathcal{E}$; here $\mathfrak{g}$ is the Lie algebra of a given Lie group G and α is a flat connection's 1-form in a principal G-bundle over $\mathcal{E}^\infty$. By construction, this $\mathfrak{g}$-valued horizontal form satisfies the Maurer–Cartan equation

$$\mathcal{E}_{\text{MC}} = \bigl\{ \mathrm{d}_h\alpha \doteq \tfrac{1}{2}[\alpha, \alpha] \bigr\} \tag{6.3}$$

by virtue of $\mathcal{E}$ and its differential consequences which constitute $\mathcal{E}^\infty$. System (6.3) is always gauge-invariant so that there are linear Noether's identities (6.5) between the equations; if the base manifold M^n is three-dimensional, then the Maurer–Cartan equation $\mathcal{E}_{\text{MC}}$ is Euler–Lagrange with respect to action functional (6.6). The main result of this chapter (see Theorem 9 on p. 94) is that – whenever one takes not just the bundle χ for $\mathfrak{g}$-valued 1-forms but the Whitney sum of four (infinite jet bundles over) vector bundles with prototype fibers built from $\mathfrak{g}$, $\Pi\mathfrak{g}$, $\mathfrak{g}^*$, and $\Pi\mathfrak{g}^*$ – the gauge invariance in (6.3) is captured by evolutionary vector field (6.10) with Hamiltonian $\widehat{S}$ that satisfies the classical master-equation [3, 38],

$$\mathcal{E}_{\text{CME}} = \bigl\{ \boldsymbol{i}\hbar\, \Delta\widehat{S}\big|_{\hbar=0} = \tfrac{1}{2}[\![\widehat{S}, \widehat{S}]\!] \bigr\}. \tag{6.1}$$

We notice that, by starting with the geometry of solutions to Maurer–Cartan's equation (6.3), we have constructed another object in the category of differential graded Lie algebras [74]; namely, we arrive at a setup with *zero* differential $\boldsymbol{i}\hbar\,\Delta\big|_{\hbar=0}$ and Lie (super-)algebra structure defined by the variational Schouten bracket $[\![\,,\,]\!]$. That geometry's genuine differential at $\hbar \neq 0$ is given by the Batalin–Vilkovisky Laplacian Δ (see [7, 8] and [63] for its definition). Let us now examine whether the standard BV-technique ([7, 8, 46], cf. [21]) can be directly applied to the case of zero-curvature representations, hence to quantum inverse scattering ([116] and [77], also [31, 37]).

It is obvious that the equations of motion $\mathcal{E}$ upon physical fields $u = \phi(x)$ co-exist with the Maurer-Cartan equations satisfied by zero-curvature representations α. The geometries of non-Abelian variational Lie algebroids and gauge algebroids [6, 61] are two manifestations of the same construction; let us stress that the respective gauge groups can be unrelated: there is the Lie group G for $\mathfrak{g}$-valued zero-curvature representations α and, on the other hand, there is a gauge group (if any, see footnote 9) for physical fields and their equations of motion $\mathcal{E} = \{\delta S_0/\delta u = 0\}$.

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[9]The class of admissible models is much wider than it may first seem; for example, the Korteweg–de Vries equation $w_t = -\frac{1}{2}w_{xxx} + 3ww_x$ is Euler–Lagrange with respect to the action functional $S_0 = \int \bigl\{ \frac{1}{2}v_xv_t - \frac{1}{4}v_{xx}^2 - \frac{1}{2}v_x^3 \bigr\}\,\mathrm{d}x \wedge \mathrm{d}t$ if one sets $w = v_x$. In absence of the model's own gauge group, its BV-realization shrinks but there remains gauge invariance in the Maurer–Cartan equation.

We recalled in section 6 that the Maurer–Cartan equation $\mathcal{E}_{\mathrm{MC}}$ itself is Euler–Lagrange with respect to functional (6.6) in the class of bundles over threefolds, cf. [2, 3, 124]. One obtains the Batalin–Vilkovisky action by extending the geometry of zero-curvature representations in order to capture Noether's identities (6.5). It is readily seen that the required set of Darboux variables consists of

- the coordinates $\mathcal{F}$ along fibres in the bundle $\mathfrak{g}^* \otimes \Lambda^2(M^3)$ for the equations $\mathcal{E}_{\mathrm{MC}}$,
- the *antifields* $\mathcal{F}^\dagger$ for the bundle $\Pi\mathfrak{g} \otimes \Lambda^1(M^3)$ which is dual to the former and which has the opposite $\mathbb{Z}_2$-valued ghost parity,[10] and also
- the *antighosts* $b^\dagger$ along fibres of $\mathfrak{g}^* \otimes \Lambda^3(M^3)$ which reproduce syzygies (6.5), as well as
- the *ghosts* b from the dual bundle $\Pi\mathfrak{g} \times M^3 \to M^3$.

The standard Koszul–Tate term in the Batalin–Vilkovisky action is then $\langle b, \boldsymbol{\partial}_\alpha^\dagger(\alpha^\dagger)\rangle$: the classical master-action for the entire model is then[11]

$$(S_0 + \langle \text{BV-terms}\rangle) + (S_{\mathrm{MC}} + \langle \text{Koszul-Tate}\rangle);$$

the respective BV-differentials anticommute in the Whitney sum of the two geometries for physical fields and flat connection $\mathfrak{g}$-forms.

The point is that Maurer–Cartan's equation (6.3) is Euler–Lagrange only if $n = 3$; however, the system $\mathcal{E}_{\mathrm{MC}}$ remains gauge invariant at all $n \geqslant 2$ but the attribution of (anti)fields and (anti)ghosts to the bundles as above becomes *ad hoc* if $n \neq 3$. We therefore propose to switch from the BV-approach to a picture which employs the four neighbours $\mathfrak{g}$, $\Pi\mathfrak{g}$, $\mathfrak{g}^*$, and $\Pi\mathfrak{g}^*$ within the master-action $\widehat{S}$. This argument is supported by the following fact [51]: let $n \geqslant 3$ for M^n, suppose $\mathcal{E}$ is nonoverdetermined, and take a finite-dimensional Lie algebra $\mathfrak{g}$, then every $\mathfrak{g}$-valued zero-curvature representation α for $\mathcal{E}$ is gauge equivalent to zero (i.e., there exists $g \in C^\infty(\mathcal{E}^\infty, G)$ such that $\alpha = \mathrm{d}_h g \cdot g^{-1}$). It is remarkable that Marvan's homological technique, which contributed with the anchor $\boldsymbol{\partial}_\alpha$ to our construction of non-Abelian variational Lie algebroids, was designed for effective inspection of the spectral

[10]The co-multiple $|\mathcal{F}\rangle$ of a $\mathfrak{g}$-valued test shift $\langle\delta\alpha|$ with respect to the $\Lambda^3(M^3)$-valued coupling $\langle\,,\,\rangle$ refers to $\mathfrak{g}^*$ at the level of Lie algebras (i.e., regardless of the ghost parity and regardless of any tensor products with spaces of differential forms). This attributes the left-hand sides of Euler–Lagrange equations $\mathcal{E}_{\mathrm{MC}}$ with $\mathfrak{g}^* \otimes \Lambda^2(M^3)$. However, we note that the pair of canonically conjugate variables would be α for $\mathfrak{g} \otimes \Lambda^1(M^3)$ and $\alpha^\dagger$ for $\Pi\mathfrak{g}^* \otimes \Lambda^2(M^3)$ whenever the Maurer–Cartan equations $\mathcal{E}_{\mathrm{MC}}$ are brute-force labelled by using the respective unknowns, that is, if the metric tensor t_{ij} is not taken into account in the coupling $\langle\delta\alpha, \mathcal{F}\rangle$.

[11]We recall that the Koszul–Tate component of the full BV-differential $\boldsymbol{D}_{\mathrm{BV}}$ is addressed in [121] by using the language of infinite jet bundles — whereas it is the BRST-component of $\boldsymbol{D}_{\mathrm{BV}}$ which we focus on in this chapter.

parameters' (non)removability at $n = 2$ but *not* in the case of higher dimensions $n \geqslant 3$ of the base M^n.

We conclude that the approach to quantisation of kinematically integrable systems is not restricted by the BV-technique only; for one can choose between the former and, e.g., flat deformation of (structures in) equation (6.1) to the quantum setup of (6.2). It would be interesting to pursue this alternative in detail towards the construction of quantum groups [31] and approach of [77, 116] to quantum inverse scattering and quantum integrable systems. This will be the subject of another research.

Bibliography

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur (1973) Method for solving the sine-Gordon equation. *Phys. Rev. Lett.*, **30**, pp. 1262–1264. doi:10.1103/PhysRevLett.30.1262.

[2] A. Achúcarro and P. K. Townsend (1986) A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories. *Phys. Lett. B*, **180**(1-2), pp. 89–92. doi:10.1016/0370-2693(86)90140-1.

[3] M. Alexandrov, A. Schwarz, O. Zaboronsky, and M. Kontsevich (1997) The geometry of the master equation and topological quantum field theory. *Internat. J. Modern Phys. A*, **12**(7), pp. 1405–1429. doi:10.1142/S0217751X97001031. `arXiv:hep-th/9502010`.

[4] S. Andrea, A. Restuccia, and A. Sotomayor (2005) The Gardner category and nonlocal conservation laws for $N = 1$ super KdV. *J. Math. Phys.*, **46**(10), pp. 103517, 11. doi:10.1063/1.2073289. `arXiv:hep-th/0504149`.

[5] A. V. Bäcklund (1881) Zur Theorie der Flächentransformationen. *Math. Ann.*, pp. 387–422.

[6] G. Barnich (2010) A note on gauge systems from the point of view of Lie algebroids. In *XXIX Workshop on Geometric Methods in Physics*, *AIP Conf. Proc.*, volume 1307. Amer. Inst. Phys., Melville, NY, pp. 7–18. `arXiv:1010.0899 [math-ph]`.

[7] I. A. Batalin and G. A. Vilkovisky (1981) Gauge algebra and quantization. *Phys. Lett. B*, **102**(1), pp. 27–31. doi:10.1016/0370-2693(81)90205-7.

[8] I. A. Batalin and G. A. Vilkovisky (1983) Quantization of gauge theories with linearly dependent generators. *Phys. Rev. D (3)*, **28**(10), pp. 2567–2582. doi:10.1103/PhysRevD.28.2567.

[9] C. Becchi, A. Rouet, and R. Stora (1976) Renormalization of gauge theories. *Ann. Physics*, **98**(2), pp. 287–321. doi:10.1016/0003-4916(76)90156-1.

[10] F. A. Berezin (1987) *Introduction to superanalysis*. Mathematical Physics and Applied Mathematics, D. Reidel Publishing Co., Dordrecht–Boston, MA.

[11] L. Bianchi (1927) *Lezioni di geometria differenziale*. Zanichelli, Pisa, 3 edition.

[12] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor'kova, I. S. Krasil'shchik, A. V. Samokhin, Y. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov (1999) *Symmetries and conservation laws for differential equations of mathematical physics*, *Translations of Mathematical Monographs*, volume 182. American Mathematical Society, Providence, RI, xiv+333 pp.

[13] L. Bonora, S. Krivonos, and A. Sorin (1996) Towards the construction of $N = 2$ supersymmetric integrable hierarchies. *Nuclear Phys. B*, **477**(3), pp. 835–854. doi:10.1016/0550-3213(96)00377-X. `arXiv:hep-th/9604165`.

[14] A. B. Borisov, M. P. Pavlov, and S. A. Zykov (2001) Proliferation scheme for Kaup-Boussinesq system. *Phys. D*, **152/153**, pp. 104–109. doi:10.1016/S0167-2789(01)00163-4.

[15] A. B. Borisov and S. A. Zykov (1998) The dressing chain of discrete symmetries and the proliferation of nonlinear equations. *Theor. Math. Phys.*, **115**(2), pp. 199–214. doi:10.1007/BF02575453.

[16] N. Bourbaki (1989) *Lie groups and Lie algebras. Chapters 1–3*. Elements of Mathematics (Berlin), Springer-Verlag, Berlin, xviii+450 pp.

[17] S. Bourque and P. Mathieu (2001) The Painlevé analysis for $N = 2$ super Korteweg-de Vries equations. *J. Math. Phys.*, **42**(8), pp. 3517–3539. doi:10.1063/1.1369641. `arXiv:math-ph/0006029`.

[18] L. J. F. Broer (1975) Approximate equations for long water waves. *Appl. Sci. Res.*, **31**(5), pp. 377–395.

[19] J. C. Brunelli and A. Das (1994) The supersymmetric two boson hierarchies. *Phys. Lett. B*, **337**(3-4), pp. 303–307. doi:10.1016/0370-2693(94)90979-2.

[20] E. Cartan (1946) *Les systèmes différentiels extérieurs et leurs applications géométriques*. Hermann, Paris.

[21] A. S. Cattaneo, P. Mnev, and N. Reshetikhin (2013) Semiclassical quantization of classical field theories. Preprint, `arXiv:1311.2490 [math-ph]`.

[22] M. Chaichian and P. P. Kulish (1987) Superconformal algebras and their relation to integrable nonlinear systems. *Phys. Lett. B*, **183**(2), pp. 169–174. doi:10.1016/0370-2693(87)90432-1.

[23] S. S. Chern and K. Tenenblat (1986) Pseudo-spherical surfaces and evolution equations. *Stud. Appl. Math.*, **74**(1), pp. 55–83.

[24] J. Cieblinski, P. Goldstein, and A. Sym (1994) On integrability of the inhomogeneous Heisenberg ferromagnet model: Examination of a new test. *J. Phys. A.: Math. Gen.*, **27**(5), p. 1645. doi:10.1088/0305-4470/27/5/028.

[25] J. Cieśliński (1993) Group interpretation of the spectral parameter in the case of nonhomogeneous, nonlinear Schrödinger system. *J. Math. Phys.*, **34**(6), pp. 2372–2384. doi:http://dx.doi.org/10.1063/1.530122.

[26] J. Cieslinski (1993) Nonlocal symmetries and a working algorithm to isolate integrable geometries. *J. Phys. A.: Math. Gen.*, **26**(5), p. L267.

[27] G. Darboux (1887) *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*. Gauthier-Villars, Paris.

[28] A. Das, W.-J. Huang, and S. Roy (1992) Zero curvature condition of OSp(2/2) and the associated supergravity theory. *Internat. J. Modern Phys. A*, **7**(18), pp. 4293–4311. doi:10.1142/S0217751X92001915.

[29] L. Degiovanni, F. Magri, and V. Sciacca (2005) On deformation of Poisson manifolds of hydrodynamic type. *Comm. Math. Phys.*, **253**(1), pp. 1–24. doi:10.1007/s00220-004-1190-8. `arXiv:nlin/0103052`.

[30] P. Deligne *et al.*, eds. (1999) *Quantum fields and strings: a course for mathematicians*, volume 1-2. AMS, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, Vol. 1: xxii+723 pp.; Vol. 2: pp. i–xxiv and 727–1501 pp.

[31] V. G. Drinfel'd (1986) Quantum groups. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **155**(Differentsialnaya Geometriya, Gruppy Li i Mekh. VIII), pp. 18–49, 193. doi:10.1007/BF01247086.

[32] B. Dubrovin (1996) Geometry of 2D topological field theories. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, *Lecture Notes in Math.*, volume 1620. Springer, Berlin, pp. 120–348. doi:10.1007/BFb0094793.

[33] B. Dubrovin (2006) On Hamiltonian perturbations of hyperbolic systems of conservation laws. II. Universality of critical behaviour. *Comm. Math. Phys.*, **267**(1), pp. 117–139. doi:10.1007/s00220-006-0021-5. `arXiv:math-ph/0510032`.

[34] B. Dubrovin, S.-Q. Liu, and Y. Zhang (2006) On Hamiltonian perturbations of hyperbolic systems of conservation laws. I. Quasi-triviality of bi-Hamiltonian perturbations. *Comm. Pure Appl. Math.*, **59**(4), pp. 559–615. doi:10.1002/cpa.20111. `arXiv:math.DG/0410027`.

[35] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov (1991) *Modern Geometry – Methods and Applications. Part I: The Geometry of Surfaces, Transformation Groups, and Fields*, *Graduate Texts in Mathematics*, volume 93. Springer, Berlin.

[36] C. Ehresmann (1953) *Structures locales*. Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 10, La Bibliothèque Nationale et Universitaire de Strasbourg, 11 pp.

[37] L. D. Faddeev and L. A. Takhtajan (1987) *Hamiltonian methods in the theory of solitons*. Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, x+592 pp.

[38] G. Felder and D. Kazhdan (2012) The classical master equation. Preprint, `arXiv:1212.1631 [math.AG]`.

[39] E. V. Ferapontov and A. Moro (2009) Dispersive deformations of hydrodynamic reductions of $(2 + 1)$D dispersionless integrable systems. *J. Phys. A.: Math. Theor.*, **42**(3), pp. 035211, 15. doi:10.1088/1751-8113/42/3/035211. `arXiv:nlin.SI/0510032`.

[40] E. V. Ferapontov, A. Moro, and V. S. Novikov (2009) Integrable equations in $2 + 1$ dimensions: deformations of dispersionless limits. *J. Phys. A.: Math. Theor.*, **42**(34), pp. 345205, 18. doi:10.1088/1751-8113/42/34/345205. `arXiv:0903.3586 [nlin.SI]`.

[41] L. Frappat, P. Sorba, and A. Sciarrino (1996) Dictionary on Lie superalgebras. Preprint, `arXiv:hep-th/9607161 [hep-th]`.

[42] W. I. Fushchich and A. G. Nikitin (1994) *Symmetries of equations of quantum mechanics*. Allerton Press, Inc., New York, xvi+465 pp.

[43] I. M. Gel'fand and L. A. Dikiĭ (1975) Asymptotic properties of the resolvent of Sturm-Liouville equations, and the algebra of Korteweg-de Vries equations. *Uspehi Mat. Nauk*, **30**(5(185)), pp. 67–100. doi:10.1070/RM1975v030n05ABEH001522.

[44] H. Goldschmidt (1967) Existence theorems for analytic linear partial differential equations. *Ann. of Math. (2)*, **86**, pp. 246–270.

[45] V. A. Golovko, I. S. Krasil'shchik, and A. M. Verbovetsky (2008) Variational Poisson-Nijenhuis structures for partial differential equations. *Theor. Math. Phys.*, **154**(2), pp. 268–282. doi:10.1007/s11232-008-0022-y. `arXiv:0812.4684 [math.DG]`.

[46] M. Henneaux and C. Teitelboim (1992) *Quantization of gauge systems*. Princeton University Press, Princeton, NJ, xxviii+520 pp.

[47] V. Hussin, A. V. Kiselev, A. O. Krutov, and T. Wolf (2010) $N = 2$ supersymmetric $a = 4$-Korteweg-de Vries hierarchy derived via Gardner's deformation of Kaup-Boussinesq equation. *J. Math. Phys.*, **51**(8), pp. 083507, 19. doi:10.1063/1.3447731. `arXiv:0911.2681 [nlin.SI]`.

[48] N. H. Ibragimov (1985) *Transformation groups applied to mathematical physics*. Mathematics and its Applications (Soviet Series), D. Reidel Publishing Co., Dordrecht, xv+394 pp.

[49] S. Igonin, P. H. M. Kersten, and I. Krasil'shchik (2003) On symmetries and cohomological invariants of equations possessing flat representations. *Diff. Geom. Appl.*, **19**(3), pp. 319–342. doi:10.1016/S0926-2245(03)00049-4. `arXiv:math/0301344 [math.DG]`.

[50] S. Igonin and J. Krasil'shchik (2002) On one-parametric families of Bäcklund transformations. In *Lie groups, geometric structures and differential equations—one hundred years after Sophus Lie (Kyoto/Nara, 1999)*, *Adv. Stud. Pure Math.*, volume 37. Math. Soc. Japan, Tokyo, pp. 99–114. `arXiv:nlin.SI/0010040`.

[51] S. A. Igonin (2003) Horizontal cohomology with coefficients, and nonlinear zero-curvature representations. *Uspekhi Mat. Nauk*, **58**(1(349)), pp. 185–186. doi:10.1070/RM2003v058n01ABEH000600.

[52] A. Karasu and A. V. Kiselev (2006) Gardner's deformations of the Boussinesq equations. *J. Phys. A: Math. Gen.*, **39**(37), pp. 11453–11460. doi:10.1088/0305-4470/39/37/008. `arXiv:nlin.SI/0603029`.

[53] A. Karasu Kalkanli, S. Y. Sakovich, and I. Yurdusen (2003) Integrability of Kersten-Krasil'shchik coupled KdV-mKdV equations: singularity analysis and Lax pair. *J. Math. Phys.*, **44**(4), pp. 1703–1708. doi:http://dx.doi.org/10.1063/1.1558903.

[54] D. J. Kaup (1975) A higher-order water-wave equation and the method for solving it. *Progr. Theoret. Phys.*, **54**(2), pp. 396–408.

[55] D. J. Kaup and A. C. Newell (1978) An exact solution for a derivative nonlinear Schrödinger equation. *J. Math. Phys.*, **19**(4), pp. 798–801.

[56] P. Kersten, I. Krasil'shchik, and A. Verbovetsky (2004) Hamiltonian operators and l^*-coverings. *J. Geom. Phys.*, **50**(1-4), pp. 273–302. doi:10.1016/j.geomphys.2003.09.010. `arXiv:math.DG/0304245`.

[57] P. Kersten and J. Krasil'shchik (2002) Complete integrability of the coupled KdV-mKdV system. In *Lie groups, geometric structures and differential equations—one hundred years after Sophus Lie (Kyoto/Nara, 1999)*, *Adv. Stud. Pure Math.*, volume 37. Math. Soc. Japan, Tokyo, pp. 151–171.

[58] P. H. M. Kersten and A. S. Sorin (2002) Bi-Hamiltonian structure of the $N = 2$ supersymmetric $\alpha = 1$ KdV hierarchy. *Phys. Lett. A*, **300**(4-5), pp. 397–406. doi:10.1016/S0375-9601(02)00852-6.

[59] A. Kiselev (2007) Algebraic properties of Gardner's deformations for integrable systems. *Theor. Math. Phys.*, **152**(1), pp. 963–976. doi:10.1007/s11232-007-0081-5. `arXiv:nlin.SI/0610072`.

[60] A. Kiselev and V. Hussin (2009) Hirota's virtual multisoliton solutions of N=2 supersymmetric Korteweg-de Vries equations. *Theor. Math. Phys.*, **159**(3), pp. 833–841. doi:10.1007/s11232-009-0071-x. `arXiv:0810.0930 [nlin.SI]`.

[61] A. V. Kiselev (2012) Homological evolutionary vector fields in Korteweg–de Vries, Liouville, Maxwell, and several other models. *J. Phys.: Conf. Ser.*, **343**(1), p. 012058. `arXiv:1111.3272 [math-ph]`.

[62] A. V. Kiselev (2012) The twelve lectures in the (non)commutative geometry of differential equations. Preprint IHÉS/M/12/13.

[63] A. V. Kiselev (2013) The geometry of variations in Batalin–Vilkovisky formalism. *J. Phys.: Conf. Ser.*, **474**(1), p. 012024. `arXiv:1312.1262 [math-ph]`.

[64] A. V. Kiselev (2013) The Jacobi identity for graded-commutative variational Schouten bracket revisited. Preprint, `arXiv:1312.4140 [math-ph]`.

[65] A. V. Kiselev and A. O. Krutov (2012) Gardner's deformations of the graded Korteweg–de Vries equations revisited. *J. Math. Phys.*, **53**(10), pp. 103511, 18. doi:10.1063/1.4754288. `arXiv:1108.2211 [nlin.SI]`.

[66] A. V. Kiselev and A. O. Krutov (2013) On the (non)removability of spectral parameters in Z_2-graded zero-curvature representations and its applications. Preprint, `arXiv:1301.7143 [math.DG]`.

[67] A. V. Kiselev and A. O. Krutov (2014) Gardner's deformations as generators of new integrable systems. *J. Phys.: Conf. Ser.*, **482**(1), p. 012021. `arXiv:1312.6941 [nlin.SI]`.

[68] A. V. Kiselev and A. O. Krutov (2014) Non-Abelian Lie algebroids over jet spaces. *J. Nonlin. Math. Phys.*, **21**(2), pp. 188–213. doi:10.1080/14029251.2014.900992. `arXiv:1305.4598 [math.DG]`.

[69] A. V. Kiselev and J. W. van de Leur (2011) Involutive distributions of operator-valued evolutionary vector fields and their affine geometry. II. In *Proc. 5th Int. workshop 'Group anlysis of differential equations and integrable systems' (June 6–10, 2010; Protaras, Cyprus)*. pp. 99–109. `arXiv:0904.1555 [math-ph]`.

[70] A. V. Kiselev and J. W. van de Leur (2011) Variational Lie algebroids and homological evolutionary vector fields. *Theor. Math. Phys.*, **167**(3), pp. 772–784. doi:10.1007/s11232-011-0061-7. `arXiv:1006.4227 [math.DG]`.

[71] A. V. Kiselev and S. Ringers (2013) A comparison of definitions for the Schouten bracket on jet spaces. In *Proc. 6th Int. workshop 'Group analysis of differential equations and integrable systems' (June 18–20, 2012; Protaras, Cyprus)*. pp. 127–141. `arXiv:1208.6196 [math.DG]`.

[72] A. V. Kiselev and T. Wolf (2006) Supersymmetric representations and integrable fermionic extensions of the Burgers and Boussinesq equations. *SIGMA Symmetry Integrability Geom. Methods Appl.*, **2**, pp. Paper 030, 19. doi:10.3842/SIGMA.2006.030. `arXiv:math-ph/0511071`.

[73] A. V. Kiselev and T. Wolf (2007) Classification of integrable super-systems using the SsTools environment. *Comput. Phys. Comm.*, **177**(3), pp. 315–328. doi:10.1016/j.cpc.2007.02.113. `arXiv:nlin.SI/0609065`.

[74] M. Kontsevich and Y. Soibelman (2009) Notes on A_∞-algebras, A_∞-categories and non-commutative geometry. In *Homological mirror symmetry*, *Lecture Notes in Phys.*, volume 757. Springer, Berlin, pp. 153–219.

[75] Y. Kosmann-Schwarzbach (2004) Derived brackets. *Lett. Math. Phys.*, **69**, pp. 61–87. doi:10.1007/s11005-004-0608-8.

[76] Y. Kosmann-Schwarzbach and F. Magri (1990) Poisson-Nijenhuis structures. *Ann. Inst. H. Poincaré Phys. Théor.*, **53**(1), pp. 35–81.

[77] B. Kostant (1979) Quantization and representation theory,. In *Proceedings of the SRC/LMS Research Symposium held in Oxford, June 28–July 15, 1977*, *London Mathematical Society Lecture Note Series*, volume 34. Cambridge University Press, Cambridge-New York, pp. v+341.

[78] I. S. Krasil'shchik and P. H. M. Kersten (2000) *Symmetries and recursion operators for classical and supersymmetric differential equations*, *Mathematics and its Applications*, volume 507. Kluwer Academic Publishers, Dordrecht, xvi+384 pp.

[79] I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov (1986) *Geometry of jet spaces and nonlinear partial differential equations*, *Advanced Studies in Contemporary Mathematics*, volume 1. Gordon and Breach Science Publishers, New York, xx+441 pp.

[80] I. S. Krasil'shchik and A. M. Vinogradov (1989) Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations. *Acta Appl. Math.*, **15**(1-2), pp. 161–209. doi:10.1007/BF00131935.

[81] S. Krivonos, A. Sorin, and F. Toppan (1995) On the super-NLS equation and its relation with the $N = 2$ super-KdV equation within the coset approach. *Phys. Lett. A*, **206**(3-4), pp. 146–152. doi:10.1016/0375-9601(95)00651-I.

[82] B. A. Kupershmidt (1983) Deformations of integrable systems. *Proc. Roy. Irish Acad. Sect. A*, **83**(1), pp. 45–74.

[83] B. A. Kupershmidt (1986) Super long waves. *Mech. Res. Comm.*, **13**(1), pp. 47–51. doi:10.1016/0093-6413(86)90011-X.

[84] P. Labelle and P. Mathieu (1991) A new $N = 2$ supersymmetric Korteweg-de Vries equation. *J. Math. Phys.*, **32**(4), pp. 923–927. doi:10.1063/1.529351.

[85] C. A. Laberge and P. Mathieu (1988) $N = 2$ superconformal algebra and integrable O(2) fermionic extensions of the Korteweg-de Vries equation. *Phys. Lett. B*, **215**(4), pp. 718–722. doi:10.1016/0370-2693(88)90048-2.

[86] D. Leites (1985) Lie superalgebras. *Journal of Soviet Mathematics*, **30**(6), pp. 2481–2512. doi:10.1007/BF02249121.

[87] D. A. Leĭtes (1980) Introduction to the theory of supermanifolds. *Uspekhi Mat. Nauk*, **35**(1(211)), pp. 3–57, 255. doi:10.1070/RM1980v035n01ABEH001545.

[88] D. Levi, A. Sym, and G. Z. Tu (1990) A working algorithm to isolate integrable surfaces in $\mathbb{E}^3$. Preprint DF INFN 761, Roma, Oct. 10,.

[89] S. Lie (1891) *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*. B. G. Teubner, Leipzig.

[90] S. Lie (1896) *Geometrie der Berührungstransformationen*. B. G. Teubner, Leipzig.

[91] F. Magri (1978) A simple model of the integrable Hamiltonian equation. *J. Math. Phys.*, **19**(5), pp. 1156–1162. doi:10.1063/1.523777.

[92] Y. Manin (1985) Holomorphic supergeometry and Yang-Mills superfields. *Journal of Soviet Mathematics*, **30**(2), pp. 1927–1975. doi:10.1007/BF02105859.

[93] M. Marvan (1997) A direct procedure to compute zero-curvature representations. The case $\mathfrak{sl}_2$. In *The International Conference on Secondary Calculus and Cohomological Physics (Moscow, 1997)*. Diffiety Inst. Russ. Acad. Nat. Sci., Pereslavl' Zalesskiy, p. 9 pp. (electronic).

[94] M. Marvan (2002) On the horizontal gauge cohomology and nonremovability of the spectral parameter. *Acta Appl. Math.*, **72**(1-2), pp. 51–65. doi:10.1023/A:1015218422059.

[95] M. Marvan (2004) Reducibility of zero curvature representations with application to recursion operators. *Acta Appl. Math.*, **83**(1-2), pp. 39–68. doi:10.1023/B:ACAP.0000035588.67805.0b. `arXiv:nlin/0306006 [nlin.SI]`.

[96] M. Marvan (2009) Sufficient set of integrability conditions of an orthonomic system. *Found. Comput. Math.*, **9**(6), pp. 651–674. doi:10.1007/s10208-008-9039-8. `arXiv:nlin/0605009 [nlin.SI]`.

[97] M. Marvan (2010) On the spectral parameter problem. *Acta Appl. Math.*, **109**(1), pp. 239–255. doi:10.1007/s10440-009-9450-4. `arXiv:0804.2031 [nlin.SI]`.

[98] P. Mathieu (1988) Supersymmetric extension of the Korteweg-de Vries equation. *J. Math. Phys.*, **29**(11), pp. 2499–2506. doi:10.1063/1.528090.

[99] P. Mathieu (2001) Open problems for the super KdV equations. In *Bäcklund and Darboux transformations. The geometry of solitons (Halifax, NS, 1999)*, *CRM Proc. Lecture Notes*, volume 29. Amer. Math. Soc., Providence, RI, pp. 325–334. `arXiv:math-ph/0005007`.

[100] A. V. Mikhailov, A. B. Shabat, and V. V. Sokolov (1991) The symmetry approach to classification of integrable equations. In *What is integrability?*. Springer Ser. Nonlinear Dynam., Springer, Berlin, pp. 115–184.

[101] R. M. Miura (1968) Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *J. Math. Phys.*, **9**, pp. 1202–1204. doi:10.1063/1.1664700.

[102] R. M. Miura, C. S. Gardner, and M. D. Kruskal (1968) Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. *J. Math. Phys.*, **9**, pp. 1204–1209.

[103] T. Miwa, M. Jimbo, and E. Date (2000) *Solitons: Differential equations, symmetries and infinite-dimensional algebras*, *Cambridge Tracts in Mathematics*, volume 135. Cambridge University Press, Cambridge, x+108 pp.

[104] G. Monge (1819) *Application de l'analyse a la géométrie*. Bachelier, Paris, 5 edition.

[105] Y. Nutku and M. V. Pavlov (2002) Multi-Lagrangians for integrable systems. *J. Math. Phys.*, **43**(3), pp. 1441–1459. doi:10.1063/1.1427765. arXiv:hep-th/0108214.

[106] P. J. Olver (1986) *Applications of Lie groups to differential equations*, *Graduate Texts in Mathematics*, volume 107. Springer-Verlag, New York, xxvi+497 pp.

[107] L. V. Ovsiannikov (1982) *Group analysis of differential equations*. Academic Press, Inc. Harcourt Brace Jovanovich, Publishers, New York-London, xvi+416 pp.

[108] S. Palit and A. Roy Chowdhury (1996) A supersymmetric dispersive water wave equation. *J. Phys. A: Math. Gen.*, **29**(11), pp. 2853–2859. doi:10.1088/0305-4470/29/11/019.

[109] R. O. Popovych, V. M. Boyko, M. O. Nesterenko, and M. W. Lutfullin (2003) Realizations of real low-dimensional Lie algebras. *J. Phys. A: Math. Gen.*, **36**(26), pp. 7337–7360. doi:10.1088/0305-4470/36/26/309. arXiv:math-ph/0301029.

[110] Z. Popowicz (1993) The Lax formulation of the "new" $N = 2$ SUSY KdV equation. *Phys. Lett. A*, **174**(5-6), pp. 411–415. doi:10.1016/0375-9601(93)90200-J.

[111] M. Roelofs (1993) *Prolongation structures of supersymmetric systems*. Ph.D. thesis, University of Twente, Enschede, The Netherlands.

[112] S. Y. Sakovich (1995) On zero-curvature representations of evolution equations. *J. Phys. A: Math. Gen.*, **28**(10), pp. 2861–2869. doi:10.1088/0305-4470/28/10/016.

[113] S. Y. Sakovich (2004) Cyclic bases of zero-curvature representations: five illustrations to one concept. *Acta Appl. Math.*, **83**(1-2), pp. 69–83. doi:10.1023/B:ACAP.0000035589.61486.a7. arXiv:nlin/0212019.

[114] R. Sasaki (1979) Soliton equations and pseudospherical surfaces. *Nuclear Phys. B*, **154**(2), pp. 343–357. doi:10.1016/0550-3213(79)90517-0.

[115] A. Sergyeyev (2004) A simple way of making a Hamiltonian system into a bi-Hamiltonian one. *Acta Appl. Math.*, **83**(1-2), pp. 183–197. doi:10.1023/B:ACAP.0000035597.06308.8a.

[116] E. Sklyanin, L. Takhtadzhyan, and L. Faddeev (1979) Quantum inverse problem method. I. *Theor. Math. Phys*, **40**(2), pp. 688–706. doi:10.1007/BF01018718.

[117] D. C. Spencer (1969) Overdetermined systems of linear partial differential equations. *Bull. Amer. Math. Soc.*, **75**, pp. 179–239.

[118] H. Stephani (1989) *Differential equation. Their solution using symmetries.* Cambridge University Press, Cambridge, xii+260 pp.

[119] I. V. Tyutin (1975) Gauge invariance in field theory and statistical mechanics. Preprint Lebedev FIAN no. 39.

[120] A. Y. Vaintrob (1997) Lie algebroids and homological vector fields. *Russian Mathematical Surveys*, **52**(2), p. 428.

[121] A. Verbovetsky (2002) Remarks on two approaches to the horizontal cohomology: compatibility complex and the Koszul-Tate resolution. *Acta Appl. Math.*, **72**(1-2), pp. 123–131. doi:10.1023/A:1015276007463. `arXiv:math.DG/0105207`.

[122] T. Voronov (2002) Graded manifolds and Drinfeld doubles for Lie bialgebroids. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, *Contemp. Math.*, volume 315. Amer. Math. Soc., Providence, RI, pp. 131–168. doi:10.1090/conm/315/05478. `arXiv:math.DG/0105237`.

[123] G. Wilson (1981) On two constructions of conservation laws for Lax equations. *Quart. J. Math. Oxford Ser. (2)*, **32**(128), pp. 491–512. doi:10.1093/qmath/32.4.491.

[124] E. Witten (1988) Topological sigma models. *Comm. Math. Phys.*, **118**(3), pp. 411–449.

[125] T. Wolf (2004) Applications of CRACK in the classification of integrable systems. In *Superintegrability in classical and quantum systems*, *CRM Proc. Lecture Notes*, volume 37. Amer. Math. Soc., Providence, RI, pp. 283–300.

[126] T. Wolf (2009). The interactive use of SsTools. Online tutorial. `http://lie.math.brocku.ca/crack/susy/`.

[127] V. Zakharov and L. Faddeev (1971) Korteweg-de Vries equation: A completely integrable Hamiltonian system. *Functional Analysis and Its Applications*, **5**(4), pp. 280–287. doi:10.1007/BF01086739.

[128] V. Zakharov and A. Shabat (1979) Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II. *Functional Analysis and Its Applications*, **13**(3), pp. 166–174. doi:10.1007/BF01077483.

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Summary

This thesis is devoted to the construction of deformations of equations and structures in nonlinear problem mathematical physics. We consider such objects as Gardner's deformations, families of nonlocalities, and families of zero-curvature representation (ZCR) for partial differential equations (PDE). Their properties and applications are analysed in detail. A general motivation to study deformation problems in the context of nonlinear PDE is as follows. First, such deformations yield recurrence relations between the Hamiltonians of PDE hierarchies. Second, we use the deformations as starting points for construction of new integrable systems and hierarchies.

We present two solutions of Mathieu's problem of constructing Gardner's deformations for the N=2 supersymmetric a=4-Korteweg–de Vries equation (SKdV). Our *first solution* is this: On the one hand, we prove the nonexistence of supersymmetry-invariant polynomial Gardner's deformations that retract to Gardner's formulas for the Korteweg-de Vries equation (KdV) under the component reduction. On the other hand, we propose a two-step scheme for the recursive production of integrals of motion for the N=2, a=4-SKdV. First, we find a new Gardner's deformation of the Kaup–Boussinesq equation, which is contained in the bosonic limit of the super-hierarchy. This yields the recurrence relation between the Hamiltonians of the limit, whence we determine the bosonic super-Hamiltonians of the full N=2, a=4-SKdV hierarchy. Our method is applicable towards the solution of Gardner's deformation problems for other supersymmetric KdV-type systems. This solution is presented in Chapters 2 and 3 and our paper [JMP10].

To construct the alternative solution, we study the relation between Gardner's deformations and zero-curvature representations. We generalise Marvan's method for inspecting the (non)removability of spectral parameters under gauge transformations in zero-curvature representations to the case of $\mathbb{Z}_2$-graded PDE. Using this technique, we prove that the parameter in ZCR constructed by Das *et al.* for N= 2, a=4-SKdV is nonremovable. By tracking the relations between zero-curvature representations and Gardner's deformations,

we construct the *second solution* of the deformation problem for the $N=2$ supersymmetric $a=4$ Korteweg–de Vries equation. Namely, we show that the zero-curvature representation found by Das *et al.* yields a system of new nonlocal variables such that their derivatives contain the Gardner deformation for the classical KdV equation. In turn, from this system of nonlocalities we derive the Gardner's deformation for $N=2$ supersymmetric $a=4$ Korteweg–de Vries equation. This solution and generalisations of Marvan's method are described in Chapters 4 and 5 and in [JPM12] and [1301.7143]. Likewise we obtain the Gardner's deformation for the Krasil'shchik–Kersten system from a zero-curvature representation found for it by Karasu–Kalkanli *et al.*.

We consider in detail a link between deformation techniques for two types of flat structures over $\mathbb{Z}_2$-graded equations, namely, their matrix zero-curvature representations and the construction of their parametric families by using the Frölicher–Nijenhuis bracket formalism developed by Krasil'shchik *et al.* In particular, in Chapter 5 and [1301.7143] we illustrate the generation and elimination of parameters in such structures.

Gardner's deformations have other important applications. We re-address the problem of construction of new infinite-dimensional completely integrable systems on the basis of known ones, and we reveal a working mechanism for such transitions. By splitting the problem's solution in two steps, we explain how the classical technique of Gardner's deformations facilitates – in a regular way – making the first, nontrivial move, in the course of which the drafts of new systems are created (often, of hydrodynamic type). The other step then amounts to higher differential order extensions of symbols in the intermediate hierarchies (e.g., by using the techniques of Dubrovin *et al.* and Ferapontov *et al.*). In particular we show that Gardner's deformation from the Kaup–Boussinesq equation yields the Kaup–Newell system. This technique is described in Chapter 2 and our paper [JPCS14].

In the context of kinematic integrability, which we address first through realisations of Gardner's deformations in terms of Lie algebra-valued flat connections, we associate Hamiltonian homological evolutionary vector fields –which are the non-Abelian variational Lie algebroids' differentials– with zero-curvature representations for PDE. This result is described in Chapter 6 and our paper [JNMP14]. It relates the line of this research to the geometry of quantum inverse scattering (well know for the seminal works by Drinfel'd, Manin, and Faddeev's school including Reshetiknin *et al.*)

Thesis are based on recent peer-reviewed articles and one preprint.

[JMP10] *Hussin V., Kiselev A. V., Krutov A. O., Wolf T.* (2010) $N=2$ supersymmetric $a=4$-KdV hierarchy derived via Gardner's deformation of Kaup–Boussinesq equation, *J. Math. Phys.* **51**:8, 083507, 19 p. `arXiv:0911.2681` [nlin.SI]

[JPM12] *Kiselev A. V., Krutov A. O.* (2012) Gardner's deformations of the graded Korteweg–de Vries equations revisited, *J. Math. Phys.* **53**:10, 103511, 18 p. `arXiv:1108.2211` [nlin.SI]

[JNMP14] *Kiselev A. V., Krutov A. O.* (2014) Non-Abelian Lie algebroids over jet spaces, *J. Nonlin. Math. Phys.* **21**:2, 188-213. `arXiv:1305.4598` [math.DG]

[JPCS14] *Kiselev A. V., Krutov A. O.* (2014) Gardner's deformations as generators of new integrable systems, *J. Phys. Conf. Ser.* **482**, Proc. Int. workshop 'Physics and Mathematics of Nonlinear Phenomena' (June 22–29, 2013; Gallipoli (LE), Italy), 012021. – 6 p. `arXiv:1312.6941` [nlin.SI]

[1301.7143] *Kiselev A. V., Krutov A. O.* (2014) On the (non)removability of spectral parameters in $\mathbb{Z}_2$-graded zero-curvature representations and its applications. – 22 p. *Preprint* `arXiv:1301.7143v2` [math.DG]

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- International workshop 'Geometric Structures in Integrable Systems', Dubrovin lab., Lomonosov MSU, Moscow, Russia (October 30 – November 2, 2012) (poster).
- XX International conference for young scientists 'Lomonosov 2013', Faculty of Mathematics and Mechanics, Lomonosov MSU, Moscow, Russia (April 8–12, 2013) (seminar 'Algebraic topology and its applications' supervised by prof. V. M. Buchstaber).
- International conference 'Computer-analytical methods in control theory and mathematics physics', Sochi, Russia (May 3–10, 2013).
- Jubilee workshop 'Nonlinear Mathematical Physics: 20 Years of JNMP', Sophus Lie Centre, Nordfjørdeid, Norway (June 4–14, 2013) (poster).

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Samenvatting

Deformaties van vergelijkingen en structuren in niet-lineaire problemen uit de mathematische fysica

Dit proefschrift behandelt de constructie van deformaties van niet-lineaire partiële differentiaalvergelijkingen (PDVs) en structuren in de mathematische fysica. We beschouwen objecten zoals Gardner's deformaties, "nonlocalities", en families van kromming nul representaties bij PDVs. Een gedetailleerde analyse van hun eigenschappen en toepassingen wordt daarbij gegeven. Een algemene motivatie om deformatieproblemen in de context van niet-lineaire PDVs te bestuderen, ziet er als volgt uit. Ten eerste geven zulke deformaties aanleiding tot recurrentierelaties tussen de Hamiltonianen van PDV hierarchieën. Ten tweede gebruiken we de deformaties als startpunt bij de constructie van nieuwe integreerbare systemen en hierarchieën.

We presenteren twee oplossingen voor het probleem van Mathieu: dit vraagt naar het construeren van Gardner's deformaties voor de N=2 supersymmetrische a=4-Korteweg–De Vries vergelijking (SKdV). Onze *eerste oplossing* werkt als volgt. Enerzijds bewijzen we dat een supersymmetrie-invariante polynomiale Gardner's deformatie, die bovendien onder reductie van componenten aanleiding geeft tot de standaard Gardner formules voor de Korteweg–De Vries vergelijking, niet bestaat. Anderzijds geven we een uit twee stappen bestaande methode waarmee recursief bewegingsintegralen voor de N=2, a=4-SKdV worden geproduceerd. Hiermee vinden we een nieuwe Gardner's deformatie van de Kaup–Boussinesq vergelijking, die bevat is in de bosonische limiet van de super-hierarchie. Dit geeft aanleiding tot de recurrente betrekking tussen de Hamiltonianen van de limiet, waaruit we dan de bosonische super-Hamiltonianen van de volledige N=2, a=4-SKdV hierarchie bepalen. Onze methode is ook toepasbaar bij het mogelijk oplossen van Gardner's deformatieproblemen voor andere supersymmetrische KdV-achtige systemen. Deze oplossing wordt beschreven in de hoofdstukken 2 en 3 en eveneens in het artikel [JMP10].

Voor de constructe van een alternatieve oplossing bestuderenen we de relatie tussen Gardner's deformaties en representaties met kromming nul. We generaliseren de methode van Marvan waarmee bepaald wordt of spectrale parameters met behulp van ijktransformaties al of niet kunnen worden geëlimineerd, naar het geval van $\mathbb{Z}_2$-gegradeerde PDVs. Gebruikmakend van deze techniek bewijzen we dat de door Das *et al.* geconstrueerde parameter in representaties met kromming nul voor de N=2, a=4-SKdV niet geëlimineerd kan worden. Door de verbanden tussen representaties met kromming nul, en Gardner's deformaties langs te lopen, construeren we nu de *tweede oplossing* voor het deformatieprobleem bij de N=2 supersymmetrische a=4 Korteweg–De Vries vergelijking. Namelijk, eerst tonen we aan dat de kromming nul representatie gevonden door Das *et al.* aanleiding geeft tot een nieuw systeem van niet-locale variabelen met de eigenschap, dat hun afgeleiden de Gardner deformatie voor de klassieke KdV vergelijking bevatten. Vervolgens leiden we uit dit systeem van 'non-localities' de Gardner's deformatie af voor de N=2 supersymmetrische a=4 Korteweg–De Vries vergelijking. Deze oplossing en generalisaties van Marvan's methode staan beschreven in de hoofdstukken 4 en 5 en ook in [JPM12] en [1301.7143]. Op een zelfde manier krijgen we de Gardner's deformatie bij het Krasil'shchik-Kersten systeem vanuit een kromming nul representatie die hierbij gevonden was door Karasu-Kalkanli *et al.*

We geven een gedetailleerde beschouwing van een verband tussen deformatietechnieken voor twee types platte structuren over $\mathbb{Z}_2$-gegradeerde vergelijkingen, namelijk hun matrix kromming nul representaties en de constructie van hun parametrische families met behulp van het formalisme van het Frölicher-Nijenhuis haakje zoals ontwikkeld door Krasil'shchik *et al.* In het illustreren we het toevoegen en elimineren van parameters in dergelijke structuren, dit gebeurt in Hoofdstuk 5 en in [1301.7143].

Gardner's deformaties hebben nog andere belangrijke toepassingen. We beschouwen opnieuw het probleem van het construeren van nieuwe oneindig dimensionale volledig integreerbare systemen gebaseerd op reeds bekende, en we geven een werkend mechanisme hiervoor. Door de oplossing van het probleem op te delen in tweeën leggen we uit, hoe de klassieke techniek van Gardner's deformaties het mogelijk maakt om – op een reguliere manier – de eerste niet-triviale stap te zetten op weg naar de ontwikkeling van nieuwe systemen (in veel gevallen van een hydrodynamisch type). De resterende stap komt dan neer op het uitbreiden tot hogere differentiale ordes van symbolen in de tussenliggende hierarchieën (bijvoorbeeld gebruikmakend van technieken van Dubrovin *et al.* en van Ferapontov *et al.*). In het bijzonder tonen we aan dat Gardner's deformatie vanuit de Kaup-Boussinesq vergelijking resulteert in het Kaup–Newell systeem. Deze techniek wordt beschreven in Hoofdstuk 2 en in ons artikel [JPCS14].

In de context van kinematische integreerbaarheid, waarop we eerst ingaan vanuit het realiseren van Gardner's deformaties in termen van platte connecties met waarden in een Lie-algebra, associëren we Hamiltoniaanse homologische evolutionaire vectorvelden –dit zijn de niet-Abelse variationele differrentialen van een Lie-algebroid– met representaties van kromming nul voor PDVs. Dit wordt gedaan in Hoofdstuk 6 en in het artikel [JNMP14]. Het geeft een verband tussen het onderzoek in dit proefschrift en de meetkunde van quantum inverse scattering (bekend door het baanbrekende werk van Drinfel'd, Manin, en de school van Faddeev waaronder Reshetiknin *et al.*)

Краткое содержание

Деформации уравнений и структур в нелинейных задачах математической физики

Диссертация посвящена задаче построения деформаций уравнений и структур в нелинейных задачах математической физики. В работе рассмотрены такие объекты, как деформации по Гарднеру и параметрические семейства нелокальностей или представлений нулевой кривизны дифференциальных уравнений в частных производных. В данной работе детально рассмотрены их свойства и приложения. Причины для изучения таких деформаций состоят в следующем. Во-первых, с их помощью можно получить рекуррентные соотношения на сохраняющиеся гамильтонианы иерархии дифференциальных уравнений в частных производных. Во-вторых, деформации можно использовать как отправную точку для построения новых интегрируемых систем и иерархий.

В диссертации найдены два решение задачи П. Матье о построении деформации по Гарднеру для N=2 суперсимметричного a=4 уравнения Кортевега–де Фриза (СКдФ). Первое решение таково. Во-первых, доказано, что не существует полиномиальной суперсимметрично-инвариантной деформации по Гарднеру для уравнения N=2, a=4-СКдФ такой, что она содержала бы в редукции деформацию по Гарднеру для классического уравнения Кортевега–де Фриза (КдФ). Но в то же время, построена новая деформация по Гарднеру для уравнения Каупа–Буссинеска. Уравнение Каупа–Буссинеска содержится в иерархии бозоного предела уравнения N=2, a=4-СКдФ. Деформация по Гарднеру уравнения Каупа–Буссинеска задаёт рекуррентные соотношения между гамильтонианами бозонного предела, которые, в свою очередь, определяют супергамильтонианы уравнения N=2, a=4-СКдФ. Рассмотренный метод также применим к решению задачи деформации других суперсимметричных уравнений КдФ-типа. Первому решению задачи П. Матье посвящены главы 2 и 3 и

работа [JMP10].

Построению альтернативного решения предшествует детальное изучение связи между деформациями по Гарднеру и представлениями нулевой кривизны. В работе обобщен на $\mathbb{Z}_2$-градуированный случай разработанный М. Марваном метод проверки (не)устранимости параметра относительно калибровочных преобразований в представлениях нулевой кривизны дифференциальных уравнений в частных производных. Используя этот метод, мы доказали, что параметр в построенном А. Дасом и др. представлении нулевой кривизны для уравнения $N{=}2$, $a{=}4$-СКдФ неустраним калибровочными преобразованиями. Прослеживая связь между представлениями нулевой кривизны и деформациями по Гарднеру, мы строим второе решение задачи П. Матье. Показано, что представление нулевой кривизны, найденное А. Дасом и др., задает систему новых нелокальных переменных — такую, что она содержит в редукции деформацию по Гарднеру уравнения КдФ. На основе этой системы нелокальных переменных построена деформация по Гарднеру для уравнения $N{=}2$, $a{=}4$-СКдФ. Это решение и обобщение метода М. Марвана описано в главах 4 и 5 данной диссертации, статье [JMP12] и препринте [1301.7143]. Использую аналогичную технику, нам также удалось построить деформацию по Гарднеру для системы уравнений Красильщика–Керстена на основе его представления нулевой кривизны, найденного Карасу–Калканлы и др.

В главе 5 и препринте [1301.7143] рассмотрены соотношения между техниками деформации двух типов плоских структур над $\mathbb{Z}_2$-градуированными дифференциальными уравнениями в частных производных: их матричными представлениями нулевой кривизны и параметрическими семействами накрытий, деформируемых скобкой Фрёлихера–Нийенхейнса (как описано И. С. Красильщиком и др.). В частности, в главе 5 и препринте [1301.7143] мы иллюстрируем процедуру порождения и ликвидации параметров в подобных структурах.

Деформации по Гарднеру имеют также и другие важные применения: например, при решении задачи построения новых интегрируемых систем. Разбивая решение данной проблемы на два этапа, мы указываем, как классические деформации по Гарднеру помогают в её решении. На первом шаге деформации по Гарднеру могут служить источником “заготовок” новых систем (обычно — гидродинамического типа). На этапе расширения символов промежуточных иерархий могут быть использованы подходы, разработанные Дубровиным, Ферапонтовым и др. В частности, используя описанный метод, из деформации по Гарднеру уравнения Каупа–Буссинеска можно получить систему уравнений Каупа–Ньюэлла. Эти результаты приведены в главе 2 данной диссертации и работе [JPCS14].

В главе 6 установлено соответствие между гамильтоновыми гомологическими эволюционными векторными полями, реализующими структуру неабелевых вариационных алгеброидов Ли (и соответствующих дифференциалов) на пространствах бесконечных струй суперрасслоений, — и, с другой стороны, представлениями нулевой кривизны с коэффициентами в заданной алгебре Ли для дифференциальных уравнений в частных производных. Эти результаты приведены в статье [JNMP14]; они связывают проведённое выше исследование с геометрией квантового метода обратной задачи рассеяния (в том виде, в котором он известен из основополагающих работ В. Г. Дринфельда, Ю. И. Манина и школы Л. Д. Фаддеева, в частности, Н. Ю. Решетихина).

Главы данной диссертации основаны на следующих статьях в международных рецензируемых журналах и одном препринте.

[JMP10] *Hussin V., Kiselev A. V., Krutov A. O., Wolf T.* (2010) N=2 supersymmetric a=4-KdV hierarchy derived via Gardner's deformation of Kaup–Boussinesq equation, *J. Math. Phys.* **51**:8, 083507, 19 с. `arXiv:0911.2681` [nlin.SI]

[JMP12] *Kiselev A. V., Krutov A. O.* (2012) Gardner's deformations of the graded Korteweg–de Vries equations revisited, *J. Math. Phys.* **53**:10, 103511, 18 с. `arXiv:1108.2211` [nlin.SI]

[JNMP14] *Kiselev A. V., Krutov A. O.* (2014) Non-Abelian Lie algebroids over jet spaces, *J. Nonlin. Math. Phys.* **21**:2, 188-213. `arXiv:1305.4598` [math.DG]

[JPCS14] *Kiselev A. V., Krutov A. O.* (2014) Gardner's deformations as generators of new integrable systems, *J. Phys. Conf. Ser.* **482**, Proc. Int. workshop 'Physics and Mathematics of Nonlinear Phenomena' (June 22–29, 2013; Gallipoli (LE), Italy), 012021. – 6 с. `arXiv:1312.6941` [nlin.SI]

[1301.7143] *Kiselev A. V., Krutov A. O.* (2014) On the (non)removability of spectral parameters in $\mathbb{Z}_2$-graded zero-curvature representations and its applications. – 22 с. *Preprint* `arXiv:1301.7143v2` [math.DG]

Полученные результаты были доложены диссертантом на следующих конференциях:

- Третья международная конференция 'Nonlinear Waves — Theory and Applications', Пекин, КНР (12–15 июня 2013 г.).

- Международная конференция 'Supersymmetries & Quantum Symmetries' – SQS'2013, ОИЯИ, Дубна, РФ (29 июля – 3 августа 2013 г.).

- Международная конференция 'Geometric Structures in Integrable Systems', МГУ им. Ломоносова, Москва, РФ (30 октября – 2 ноября 2012 г.) (постер).

- Международная научная конференция студентов, аспирантов и молодых учёных “Ломоносов-2014” МГУ им. Ломоносова, Москва, РФ (8–12 апреля 2013 г.) (семинар “Алгебраическая топология и её приложения“, рук. чл.-корр. проф. д.ф.-м.н. В. М. Бухштабер).

- Международная конференция ‘Computer-analytical methods in control theory and mathematics physics’, Сочи, РФ (3–10 мая 2013 г.).

- Юбилейная конференция ‘Nonlinear Mathematical Physics: 20 Years of JNMP’, Sophus Lie Centre, Nordfjørdeid, Норвегия (4–14 июня 2013 г.) (постер).

Часть данного исследование выполнена во время стажировки диссертанта в математическом институте университета г. Утрехта (Нидерланды) в 2009 г.

Biography

Andrey Olegovich Krutov was born on 23 September 1986 in Shuya, Russia. He graduated with cum laude from Ivanovo State Power University (ISPU, Ivanovo, Russia) in 2009 specialising in Pure and Applied Mathematics. In 2009 he started his Ph.D. research in Mathematics in Ivanovo State Power University and he has also been working as teaching assistant at Department of Higher Mathematics (ISPU) since that year.

Andrey Krutov attend three Diffiety Schools in 2008, 2009 (Kostroma, Russia) and 2012 (Gdynia, Poland). He attend also school on geometry of partial differential equations in 2013 (Kouty nad Desnou, Czech Republic). He continues his Ph.D. research at Johann Bernoulli Institute for Mathematics and Computer Science since February 2014 under supervision of dr. A. V. Kiselev.